Physics 581: Open Quantum Systems

Problem Set #1: Solutions

Problem 1:

(a) Consider a statistical mixture

$$\hat{\rho} = P_+ |\uparrow_z\rangle\langle\uparrow_z| + P_- |\downarrow_z\rangle\langle\downarrow_z|$$

where $P_\pm = \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{2}}\right)$

Density matrix $\rho_{ij} = \langle i|\hat{\rho}|j\rangle$

In basis $|\pm_z\rangle$ $\hat{\rho} \doteq_z \frac{1}{2}\begin{pmatrix} 1+\frac{1}{\sqrt{2}} & 0 \\ 0 & 1-\frac{1}{\sqrt{2}} \end{pmatrix}$

In basis $\left(|\updownarrow_x\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle \pm |\downarrow_z\rangle)\right)$

$$\hat{\rho} \doteq_x \begin{pmatrix} \langle\uparrow_x|\hat{\rho}|\uparrow_x\rangle & \langle\uparrow_x|\hat{\rho}|\downarrow_x\rangle \\ \langle\downarrow_x|\hat{\rho}|\uparrow_x\rangle & \langle\downarrow_x|\hat{\rho}|\downarrow_x\rangle \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 \end{pmatrix}$$

Note: In this basis the density operator has off-diagonal elements. Nonetheless, it is a mixed state:

$$\mathrm{Tr}(\hat{\rho}^2) = \frac{3}{4}$$

The Bloch vector can be seen immediately from the form in the z-basis.

$$\mathrm{Tr}(\hat{\rho}\,\hat{\sigma}_x) = \mathrm{Tr}(\hat{\rho}\,\hat{\sigma}_y) = 0$$

$$\mathrm{Tr}(\hat{\rho}\,\hat{\sigma}_z) = P_{\uparrow_z} - P_{\downarrow_z} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \boxed{\vec{Q} = \frac{1}{\sqrt{2}}\vec{e}_z} \qquad \text{mixed state } |\vec{Q}| < 1$$

(c) Now we have a state

$$\hat{\rho} = \frac{1}{2}|\uparrow_{n_1}\rangle\langle\uparrow_{n_1}| + \frac{1}{2}|\uparrow_{n_2}\rangle\langle\uparrow_{n_2}|$$

where $|\uparrow_n\rangle\langle\uparrow_n| = \frac{1}{2}(\hat{\mathbb{1}} + \vec{e}_n\cdot\hat{\vec{\sigma}})$ from Prob 1

$$\vec{e}_{n_{1/2}} = \frac{1}{\sqrt{2}}(\vec{e}_z \pm \vec{e}_x)$$

$$\Rightarrow \hat{\rho} = \frac{1}{2}\hat{\mathbb{1}} + \frac{1}{4}(\vec{e}_{n_1} + \vec{e}_{n_2})\cdot\hat{\vec{\sigma}}$$

$$= \frac{1}{2}\hat{\mathbb{1}} + \frac{1}{4}\left(\frac{2}{\sqrt{2}}\vec{e}_z\right)\cdot\hat{\vec{\sigma}}$$

$$= \frac{1}{2}\left(\hat{\mathbb{1}} + \frac{1}{\sqrt{2}}\vec{e}_z\right)\cdot\vec{\sigma} \underset{z}{=} \begin{bmatrix} \frac{1}{2}(1+\frac{1}{\sqrt{2}}) & 0 \\ 0 & \frac{1}{2}(1-\frac{1}{\sqrt{2}}) \end{bmatrix}$$

Same as $\hat{\rho}$ in part (b)!

Moral of the story: The ensemble decomposition is not unique. In fact, we can take any density matrix for a two-level system, described uniquely in terms of its Bloch vector $\vec{Q}$ and decompose it in terms of an ensemble of any two pure states described by unit vector $\vec{e}_n$ with probability P_n if $\vec{Q} = P_{n_1}\vec{e}_{n_1} + P_{n_2}\vec{e}_{n_2}$.

(c) Two statistical mixtures

$$\hat{\rho}_1 = \sum_n p_n |\hat{+}_n\rangle\langle\hat{+}_n|$$

$$\hat{\rho}_2 = \sum_m q_m |\hat{+}_m\rangle\langle\hat{+}_m|$$

Aside: $|+_n\rangle\langle+_n| = \frac{1}{2}(\hat{\mathbb{1}} + \hat{\sigma}_n)$ $\left(\begin{array}{l}\text{where} \\ \hat{\sigma}_n = \vec{e}_n \cdot \hat{\vec{\sigma}}\end{array}\right)$

Projector

$$\Rightarrow \hat{\rho}_1 = \underbrace{\left(\sum_n p_n\right)}_{=1} \frac{1}{2}\hat{\mathbb{1}} + \frac{1}{2}\underbrace{\left(\sum_n p_n \vec{e}_n\right)}_{\vec{Q}_1} \cdot \hat{\vec{\sigma}}$$

$$\Rightarrow \hat{\rho}_1 = \frac{1}{2}\hat{\mathbb{1}} + \frac{1}{2}\vec{Q}_1 \cdot \hat{\vec{\sigma}}$$

Similarly $\hat{\rho}_2 = \frac{1}{2}\hat{\mathbb{1}} + \frac{1}{2}\vec{Q}_2 \cdot \hat{\vec{\sigma}}$

where $\vec{Q}_2 = \sum_m q_m \vec{e}_n$

Thus $\hat{\rho}_1 = \hat{\rho}_2 \iff \vec{Q}_1 = \vec{Q}_2$

Problem 2: Ambiguity of ensemble decomposition

Let $\hat{\rho}_1 = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, $\hat{\rho}_2 = \sum_j q_j |\phi_j\rangle\langle\phi_j|$

Prove $\hat{\rho}_1 = \hat{\rho}_2$ iff $\sqrt{q_j}\,|\phi_j\rangle = \sum_i U_{ji} \sqrt{p_i}\, |\psi_i\rangle$

where U_{ji} are elements of unitary matrix.

Proof:

For convenience, define $|\bar{\phi}_j\rangle \equiv \sqrt{q_j}\,|\phi_j\rangle$

$|\bar{\psi}_i\rangle \equiv \sqrt{p_i}\,|\psi_i\rangle$

so $\langle\bar{\phi}_j|\bar{\phi}_j\rangle = q_j$ $\quad\langle\bar{\psi}_i|\bar{\psi}_i\rangle = p_i$

(1) Assume $|\bar{\phi}_j\rangle = \sum_i U_{ji} |\bar{\psi}_i\rangle$ $\quad U_{ji}$ elements of unitary matrix

Consider $\hat{\rho}_2 = \sum_j |\bar{\phi}_j\rangle\langle\bar{\phi}_j| = \sum_{ijk} U^*_{jk} U_{ji} |\bar{\psi}_i\rangle\langle\bar{\psi}_k|$

Aside: $(U_{jk})^* = U^\dagger_{kj}$

$$\Rightarrow \hat{\rho}_2 = \sum_{ik} \underbrace{\left(\sum_j U^\dagger_{kj} U_{ji}\right)}_{\delta_{ik}} |\bar{\psi}_i\rangle\langle\bar{\psi}_k|$$

$$\Rightarrow \hat{\rho}_2 = \sum_i |\bar{\psi}_i\rangle\langle\bar{\psi}_i| = \hat{\rho}_1 \quad ✓$$

(ii) Now assume $\hat{\rho}_1 = \hat{\rho}_2 \equiv \hat{\rho}$

$\hat{\rho}$ being a Hermitian operator can be diagonalized

$$\Rightarrow \quad \hat{\rho} = \sum_\alpha \lambda_\alpha |e_\alpha\rangle\langle e_\alpha|$$

where $\begin{cases} \sum_\alpha \lambda_\alpha = 1 & \text{with } \lambda_\alpha \text{ real}, \quad 0 \le \lambda_\alpha \le 1 \\ \langle e_\alpha | e_\beta \rangle = \delta_{\alpha\beta} \end{cases}$

Let $|\bar{e}_\alpha\rangle = \sqrt{\lambda_\alpha}\, |e_\alpha\rangle \quad \Rightarrow \quad \hat{\rho} = \sum_\alpha |\bar{e}_\alpha\rangle\langle \bar{e}_\alpha|$

$$\Rightarrow \quad \sum_i |\Psi_i\rangle\langle\Psi_i| = \sum_\alpha |\bar{e}_\alpha\rangle\langle\bar{e}_\alpha| = \sum_j |\Phi_j\rangle\langle\Phi_j|$$

we seek the relationship between $\{|\Psi_i\rangle\}$ and $\{|\Phi_j\rangle\}$

<u>First note</u> $\{|e_\alpha\rangle\}$ form a basis for the Hilbert space (with $\lambda_\alpha = 0$ for those vectors not in $\hat{\rho}$)

$$\Rightarrow \quad |\Psi_i\rangle = \sum_\alpha |e_\alpha\rangle\langle e_\alpha|\Psi_i\rangle = \sum_\alpha |\bar{e}_\alpha\rangle \frac{\langle e_\alpha|\Psi_i\rangle}{\sqrt{\lambda_\alpha}}$$

$$= \sum_\alpha M_{i\alpha} |\bar{e}_\alpha\rangle$$

where $M_{i\alpha} = \dfrac{\langle e_\alpha|\Psi_i\rangle}{\sqrt{\lambda_\alpha}}$

Now: $\sum_i M_{i\alpha} M^*_{i\beta} = \sum_i \frac{\langle e_\alpha | \Psi_i \rangle \langle \Psi_i | e_\beta \rangle}{\sqrt{\lambda_\alpha \lambda_\beta}}$

$$= \frac{1}{\sqrt{\lambda_\alpha \lambda_\beta}} \langle e_\alpha | \left(\sum_i |\Psi_i\rangle\langle\Psi_i| \right) | e_\beta \rangle$$

$$= \frac{1}{\sqrt{\lambda_\alpha \lambda_\beta}} \langle e_\alpha | \hat{\rho} | e_\beta \rangle = \frac{\lambda_\alpha \delta_{\alpha\beta}}{\sqrt{\lambda_\alpha \lambda_\beta}} = \delta_{\alpha\beta}$$

$\Rightarrow$ When arranged in a matrix, the columns of $M_{i\alpha}$ are orthonormal

(Subtle point: $M_{i\alpha}$ need not be square here, since # of pure states in the $\{|\Psi_i\rangle\}$ need not be the dimension of Hilbert space. However, we can always append extra columns in the orthogonal space to make $M_{i\alpha}$ unitary.)
Formally the matrix M is a "partial isometry"

Thus since $|\Psi_i\rangle = \sum_\alpha M_{i\alpha} |\bar{e}_\alpha\rangle$

$$|\Phi_j\rangle = \sum_\beta N_{j\beta} |\bar{e}_\beta\rangle$$

$$|\Phi_j\rangle = \sum_i U_{ji} |\Psi_i\rangle$$

where $U = NM^\dagger$ q.e.d.

Problem 3: Inhomogeneous Broadening

We consider a set of spins in an inhomogenous bias field, so that the detuning see locally by a spin is uncertain. We consider the probability distribution of detunings to be Gaussian

$$P(\Delta) = \frac{1}{\sqrt{2\pi\delta^2}} e^{-\frac{(\Delta-\Delta_0)^2}{2\delta^2}}$$

If we look at Rabi flopping (spin magnetic resonance),

$$P(\uparrow_z, t) = \int_{-\infty}^{\infty} d\Delta \; P(\Delta)\, P(\uparrow_z, t | \Omega, \Delta)$$

Where $P(\uparrow_z, t|\Omega, \Delta) = \frac{\Omega^2}{\Omega^2_{tot}} \left(\frac{1-\cos(\Omega_{tot}t)}{2}\right)$ $\qquad \Omega_{tot} = \sqrt{\Omega^2 + \Delta^2}$

Rabi oscillations for a fixed Ω + Ω

This integral cannot be done analytic, in general. We can solve this approximately when $\delta \ll \Omega_{tot}(\Delta_0)$.

$$P(\uparrow_z, t) = \frac{1}{2\sqrt{2\pi\delta^2}} \int_{-\infty}^{\infty} d\Delta \; e^{-\frac{(\Delta-\Delta_0)^2}{2\delta^2}} \left[\frac{1 - \cos(\sqrt{\Omega^2 + (\Delta_0 + (\Delta-\Delta_0))^2}\, t)}{\Omega^2 + (\Delta_0 + (\Delta-\Delta_0))^2}\right]$$

Let $\xi = \Delta - \Delta_0 \qquad |\xi| \ll \Delta_0, \Omega$

- $\cos(\Omega_{tot}t) \simeq \cos(\sqrt{\Omega^2 + \Delta_0^2 + 2\xi\Delta_0}\, t) \approx \cos\left(\sqrt{\Omega^2+\Delta_0^2}\, t + \frac{\xi\Delta_0}{\sqrt{\Omega_0^2+\Delta^2}} t\right)$

- Aside: $\int_{-\infty}^{\infty} d\xi \; e^{-\xi^2/2\delta^2} e^{\pm i a\xi} = e^{-\frac{a^2\delta^2}{2}} \sqrt{2\pi a^2}$

$$\Rightarrow \boxed{P(\uparrow_z, t) \approx \frac{\Omega^2}{\Omega^2+\Delta_0^2}\left(\frac{1 - \cos(\sqrt{\Omega^2+\Delta_0^2}\, t)\, e^{-\frac{\Delta_0^2\delta^2t^2}{2(\Omega^2+\Delta_0^2)}}}{2}\right)}$$

This has the form $P(\uparrow_z, t) = \frac{1}{2}\frac{\Omega^2}{\Omega^2_{tot}}\left(1 - \cos\Omega_{tot}t \; e^{-\frac{t^2}{2T_2^{*2}}}\right)$

where $\Omega_{tot} = \sqrt{\Omega^2 + \Delta^2}$, $\quad \frac{1}{T_2^*} = \frac{\delta|\Delta_0|}{\Omega_{tot}}$ $\quad$ <u>Inhomogeneous linewidth</u>

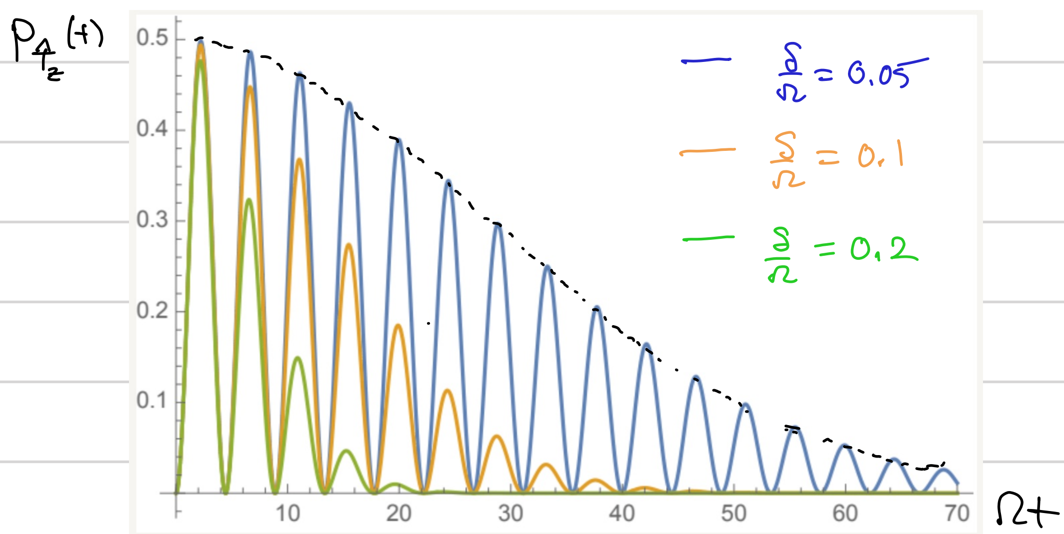
When $|\Delta_0| \gg \Omega$, $\Omega_{tot} \sim |\Delta_0| \implies \frac{1}{T_2^*} = \delta$

When $|\Delta_0| \ll \Omega$, $\Omega_{tot} \sim \Omega \implies \frac{1}{T_2^*} \simeq \delta\left(\frac{|\Delta_0|}{\Omega}\right) \to 0$ as $\frac{|\Delta_0|}{\Omega} \to 0$

(c) In dimensionless form $P(\uparrow_z, t) = \frac{1}{2\sqrt{1+(\Delta_0/\Omega)^2}}\left[1 - \cos\left(\sqrt{1+\frac{\Delta_0^2}{\Omega^2}}\,\Omega t\right) e^{-\left(\frac{\delta}{\Omega}\right)^2 \frac{(\Delta_0/\Omega)^2}{1+\left(\frac{\Delta_0}{\Omega}\right)^2}(\Omega t)^2}\right]$

Taking $\frac{|\Delta_0|}{\Omega} = 1$ $\quad P(\uparrow_z, t) = \frac{1}{2\sqrt{2}}\left[1 - \cos(\sqrt{2}\,\Omega t)\right] e^{-\left(\frac{\delta}{\Omega}\right)^2 \frac{(\Omega t)^2}{2}}$

Plotted below are examples for $\frac{\delta}{\Omega} = 0.05, .1, .2$

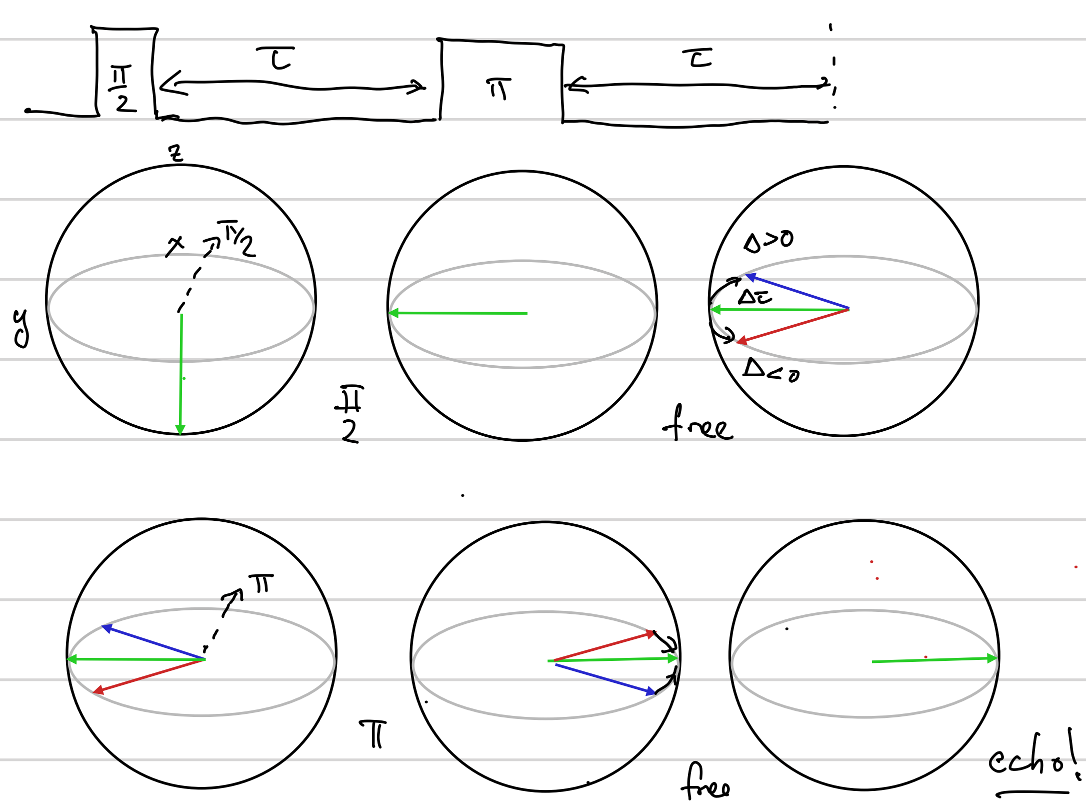


We see here the dephasing of the oscillations with a Gaussian envelope due to the Gaussian spread in detunings

(d) Spin Echo: Time reversing inhomogeneous (but coherent) evolution

Consider the following sequence: fast $\frac{\pi}{2}$-pulse about x ~ free evolution for time τ

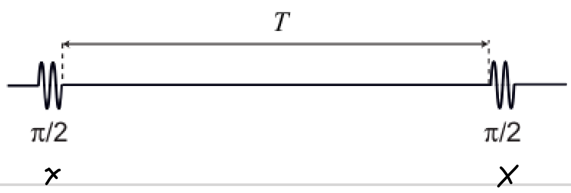
π-pulse about x ~ free evolution for time τ



The π-pulse flips the spins so that the the spins refocus at time τ

Problem 4: Measuring T_2 times via a Ramsey interferometer

(a) The standard Ramsey "separated zone" pulse sequence provides a method for measuring the coherence time in a quantum superposition between two orthogonal states $\{|0\rangle, |1\rangle\}$.



We seek the probability $P_0 = \langle 0|\hat{\rho}_{out}|0\rangle$ where $\hat{\rho}_{out}$ is the output state of the "inteferometer." Without decoherence,

$$\hat{\rho}_{out} = \underbrace{\hat{U}_x(\tfrac{\pi}{2})}_{\text{final pulse}}\underbrace{\hat{U}_z(\Delta t)}_{\text{free evolution}}\underbrace{\hat{U}_x(\tfrac{\pi}{2})}_{\text{first pulse}}|1\rangle\langle 1|\hat{U}_x^{\dagger}(\tfrac{\pi}{2})\hat{U}_z^{\dagger}(\Delta t)\hat{U}_x^{\dagger}(\tfrac{\pi}{2})$$

$$\Rightarrow \hat{\rho}_{out} = e^{-i\frac{\pi}{4}\hat{\sigma}_x}\hat{\rho}(T)e^{+i\frac{\pi}{4}\hat{\sigma}_x}$$ where $\hat{\rho}(T)$ = state before final pulse

$$\Rightarrow P_0(T) = \langle 0|e^{-i\frac{\pi}{4}\hat{\sigma}_x}\hat{\rho}(T)e^{i\frac{\pi}{4}\hat{\sigma}_x}|0\rangle = \left(\frac{\langle 0| - i\langle 1|}{\sqrt{2}}\right)\hat{\rho}(T)\left(\frac{|0\rangle + i|1\rangle}{\sqrt{2}}\right)$$

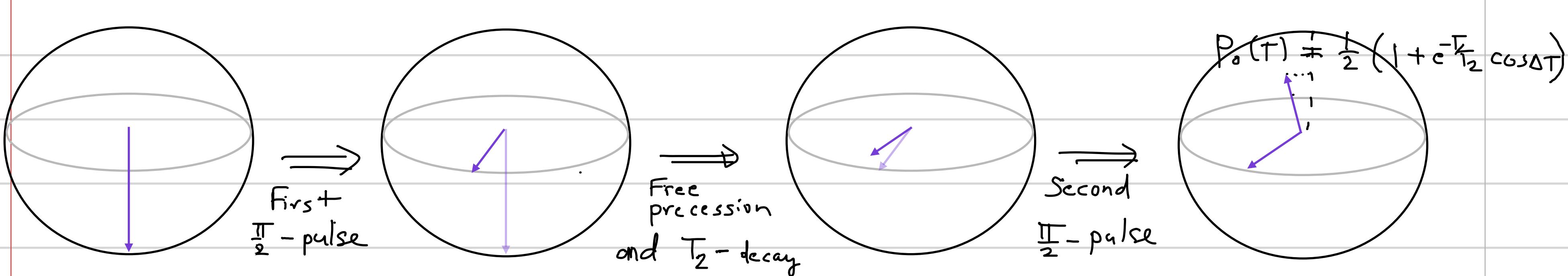
$$= \frac{1}{2}\left(\rho_{00}(T) + \rho_{11}(T) + i\left(\rho_{01}(T) - \rho_{10}(T)\right)\right) = \frac{1}{2}\left(1 + 2\,\mathrm{Im}\left(\rho_{10}(T)\right)\right)$$ ← Interferometer measures coherence

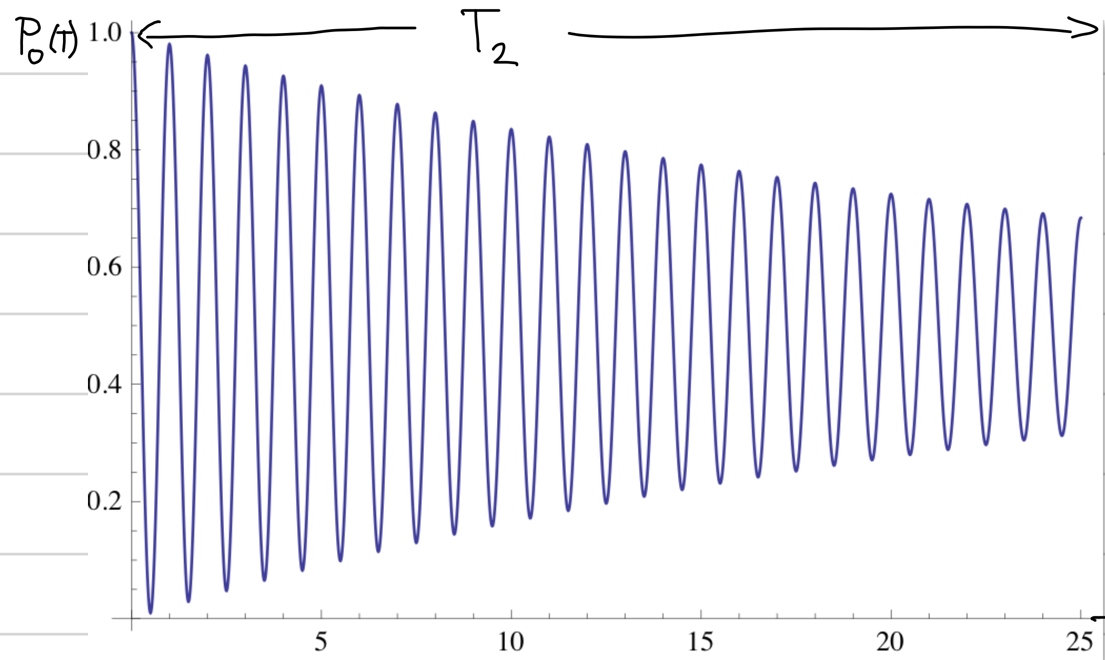
With decoherence;

$$\rho_{10}(T) = \underbrace{e^{-T/T_2}}_{T_2\text{-decay}}\underbrace{\langle 1|}_{\langle\downarrow_z|}e^{+i\frac{\Delta T}{2}\hat{\sigma}_z}\underbrace{e^{-i\frac{\pi}{4}\hat{\sigma}_x}\hat{\rho}(0)e^{i\frac{\pi}{4}\hat{\sigma}_x}}_{|\uparrow_y\rangle\langle\uparrow_y|}e^{-i\frac{\Delta T}{2}\hat{\sigma}_z}\underbrace{|0\rangle}_{|\uparrow_z\rangle} = e^{-T/T_2}\langle\downarrow_z|e^{i\frac{\Delta T}{2}\hat{\sigma}_z}|\uparrow_y\rangle\langle\uparrow_y|e^{-i\frac{\Delta T}{2}\hat{\sigma}_z}|\uparrow_z\rangle$$

$$\Rightarrow \rho_{10}(T) = \frac{i}{2}e^{-T/T_2}e^{-i\Delta T}$$

$$\Rightarrow \boxed{P_0(T) = \frac{1}{2}\left(1 + e^{-T/T_2}\cos(\Delta T)\right)}$$



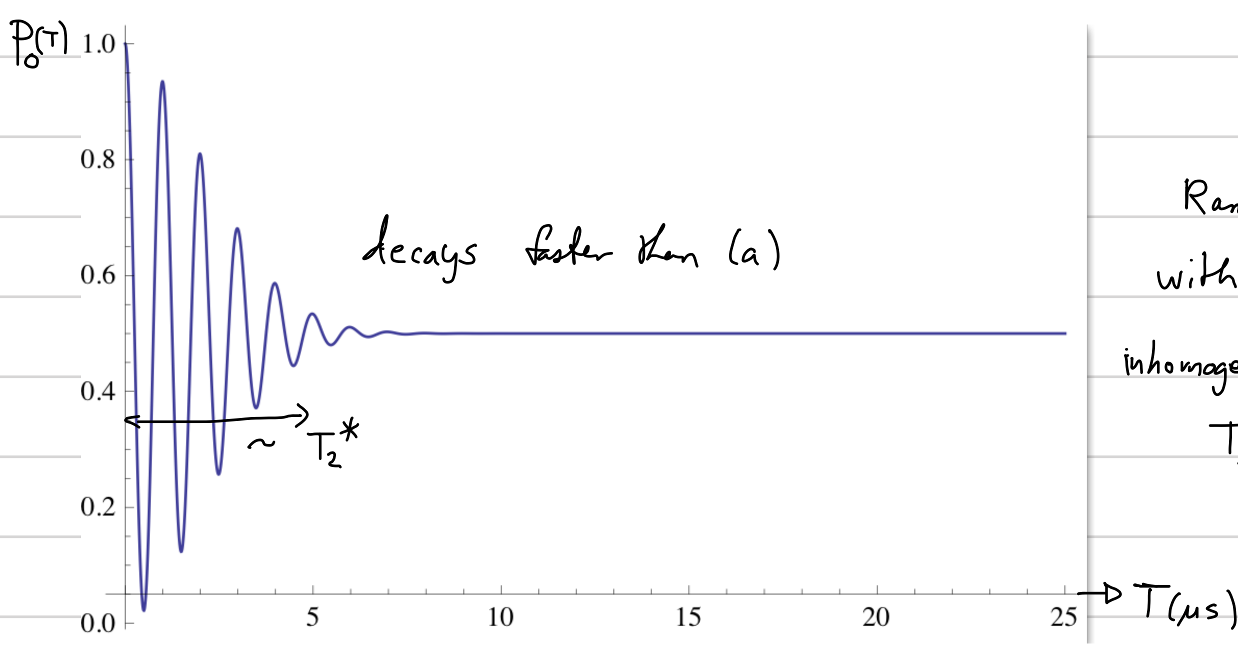


Example of decay Ramsey fringes for $\Delta/2\pi = 1$ MHz, $T_2 = 25\,\mu s$.

(b) Now we add inhomogeneous broadening – a distribution of detunings $p(\Delta) = \dfrac{e^{-\frac{(\Delta-\Delta_0)^2}{2\delta^2}}}{\sqrt{2\pi\delta^2}}$

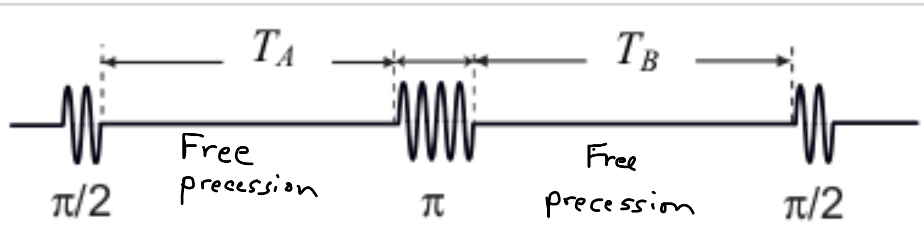
$$\Rightarrow P_0(T) = \int d\Delta\, p(\Delta)\, P_0(T,\Delta) = \int d\Delta\, p(\Delta) \left(\frac{1 + e^{-T/T_2} \cos(\Delta T)}{2} \right)$$

$$P_0(T) = \frac{1}{2}\left(1 + e^{-T/T_2}\, e^{-\frac{T^2}{2T_2^{*2}}} \cos(\Delta_0 T)\right) \quad \text{where} \quad T_2^* = \frac{1}{\delta}$$



Ramsey Two-pulse sequence with same mean detuning but inhomogeneously broadened, with a $T_2^* = 5\,\mu s < T_2 = 25\,\mu s$

(c) The Hahn spin echo sequence removes the inhomogenous brodening in Δ to first order.



"hard pulse" around y

"hard pulse" around x

"hard pulse" around y

Without decoherence or dephasing, this spin echo sequence, the probability

$$P_0 = |\langle 0| e^{-i\frac{\pi}{4}\hat{\sigma}_y} e^{-i\frac{\Delta T_B}{2}\hat{\sigma}_z} e^{-i\frac{\pi}{2}\hat{\sigma}_x} e^{-i\frac{\Delta T_A}{2}\hat{\sigma}_z} e^{-i\frac{\pi}{4}\hat{\sigma}_y}|1\rangle|^2$$

<u>Aside</u> $e^{-i\frac{\pi}{2}\hat{\sigma}_x} e^{-i\frac{\Delta T_A}{2}\hat{\sigma}_z} e^{-i\frac{\pi}{4}\hat{\sigma}_y} = e^{+i\frac{\Delta T_A}{2}\hat{\sigma}_z} e^{-i\frac{\pi}{2}\hat{\sigma}_x} e^{-i\frac{\pi}{4}\hat{\sigma}_y}$

Insert $(e^{+i\frac{\pi}{2}\hat{\sigma}_x} e^{-i\frac{\pi}{2}\hat{\sigma}_x})$ $\qquad e^{+i\frac{\pi}{2}\hat{\sigma}_x} e^{-i\frac{\pi}{2}\hat{\sigma}_x}$

$$= e^{i\frac{\Delta T_A}{2}\hat{\sigma}_z} e^{+i\frac{\pi}{4}\hat{\sigma}_y} e^{-i\frac{\pi}{2}\hat{\sigma}_x}$$

This follows from unitary transformations that rotate $\hat{\sigma}_z \to -\hat{\sigma}_z$, $\hat{\sigma}_y \to -\hat{\sigma}_y$ when we rotate about x by $180°$

$$\Rightarrow P_0 = |\langle 0| \underbrace{e^{-i\frac{\pi}{4}\hat{\sigma}_y} e^{-i\Delta(T_A - T_B)\hat{\sigma}_z/2} e^{+i\pi/4\,\hat{\sigma}_y}}_{e^{-i\Delta(T_A-T_B)\hat{\sigma}_x/2}} \underbrace{e^{-i\frac{\pi}{2}\hat{\sigma}_x}|1\rangle}_{|0\rangle}|^2$$

(Since $\hat{\sigma}_z \to \hat{\sigma}_x$ with rotation by $90°$ about y

$$\Rightarrow P_0 = \cos^2\left(\frac{\Delta(T_A - T_B)}{2}\right) = \frac{1}{2}\left(1 + \cos(\Delta(T_A - T_B))\right)$$

Including homogeneous T_2 decay: $P_0(T, \Delta, T_2) = \frac{1}{2}\left[1 + \underbrace{e^{-\frac{T_A+T_B}{T_2}}}_{\text{decay during } T_A \,\&\, T} \cos\{\Delta(T_A - T_B)\}\right]$

Including both T_2 and inhomogeneous T_2^*:

$$P_0(T_A, T_B, \Delta_0, T_2, T_2^*) = \int d\Delta\, p(\Delta, T_2^*, \Delta_0)\, P_0(T_A, T_B, \Delta, T_2)$$

$$\Rightarrow \boxed{P_0(T_A, T_B, \Delta_0, T_2, T_2^*) = \frac{1}{2}\left[1 - e^{\frac{-(T_A+T_B)}{T_2}} e^{-\frac{1}{2}\left(\frac{T_A - T_B}{T_2^*}\right)^2} \cos\{\Delta_0(T_A - T_B)\}\right]}$$

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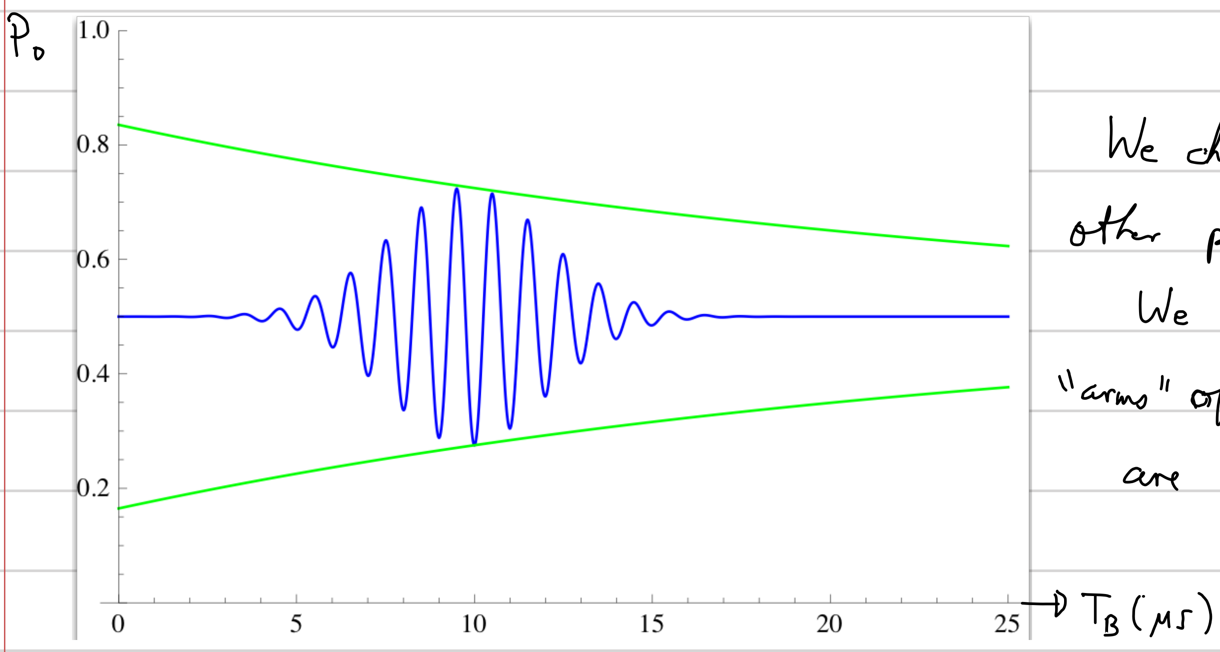

We chose here $T_A = 10\,\mu s$ – all other parameters are as in part (b).
We see the "echo" when the two "arms" of the Ramsey interferometer are near equal $T_A \sim T_B$

The Hahn echo allows us to measure the true decay time T_2 (shown here as the green exponential decay curves), removing the dephasing due to inhomogeneity.