Physics 581: Open Quantum Systems

Problem Set #2: Solutions

Problem 1: Boson Algebra

(a) Completeness of coherent states: $|\alpha\rangle = \sum_n c_n |n\rangle$, $c_n = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}}$

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = \sum_{n,m} \int \frac{d^2\alpha}{\pi} e^{-|\alpha|^2} \frac{(\alpha^*)^n (\alpha)^m}{\sqrt{n!\,m!}} |m\rangle\langle n| = \sum_{n,m} \int_0^\infty r dr\, e^{-r^2} \frac{r^{n+m}}{\sqrt{n!\,m!}} \int_0^{2\pi} \frac{d\phi}{\pi} e^{i(m-n)\phi} |m\rangle\langle n|$$

Let $\alpha = re^{i\phi}$

$$= \sum_{n,m} \int_0^\infty r dr\, e^{-r^2} \frac{r^{n+m}}{\sqrt{n!\,m!}} (2\delta_{nm}) |m\rangle\langle n| = \sum_n \frac{2}{n!} \Big(\underbrace{\int_0^\infty e^{-r^2} r^{2n+1} dr}_{n!/2} \Big) |m\rangle\langle n|$$

$$\Rightarrow \int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = \sum_{n=0}^{\infty} |n\rangle\langle n| = \hat{\mathbb{1}} \quad ✓$$

(b) Weyl-Heisenberg group:

Using $e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}+\hat{B}} e^{\frac{1}{2}[\hat{A},\hat{B}]}$ where $[\hat{A},\hat{B}]$ commute with $\hat{A}$ and $\hat{B}$

$$\hat{D}(\alpha)\hat{D}(\beta) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} e^{\beta \hat{a}^\dagger - \beta^* \hat{a}} = e^{(\alpha+\beta)\hat{a} - (\alpha+\beta)^* \hat{a}^\dagger} e^{\frac{1}{2}\underbrace{[\alpha \hat{a}^\dagger - \alpha^* \hat{a},\, \beta \hat{a}^\dagger - \beta^* \hat{a}]}_{\alpha\beta^* - \alpha^*\beta}}$$

$$\Rightarrow \boxed{\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha+\beta)\, e^{i\,\mathrm{Im}(\alpha\beta^*)}}$$

$$\boxed{\begin{aligned} \langle\alpha|\beta\rangle &= \langle 0|\hat{D}^\dagger(\alpha)\hat{D}(\beta)|0\rangle = \langle 0|\hat{D}(-\alpha)\hat{D}(\beta)|0\rangle = \langle 0|\hat{D}(\beta-\alpha)|0\rangle\, e^{-i\,\mathrm{Im}(\alpha\beta^*)} \\ &= e^{-\frac{1}{2}|\alpha-\beta|^2} e^{-i\,\mathrm{Im}(\alpha\beta^*)} \qquad \text{Note } |\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2} \to 0 \text{ when } |\alpha-\beta| \gg 1 \end{aligned}}$$

(c) $$\mathrm{Tr}\big(\hat{D}^\dagger(\alpha)\hat{D}(\beta)\big) = \int \frac{d^2\gamma}{\pi} \langle\gamma|\hat{D}^\dagger(\alpha)\hat{D}(\beta)|\gamma\rangle = \int \frac{d^2\gamma}{\pi} \langle\gamma|\hat{D}(\beta-\alpha)|\gamma\rangle\, e^{\frac{\beta\alpha^* - \alpha^*\beta}{2}}$$

Aside: $\langle\gamma|\hat{D}(\alpha-\beta)|\gamma\rangle = e^{-\frac{1}{2}|\alpha-\beta|^2} \langle\gamma| e^{(\alpha-\beta)\hat{a}^\dagger} e^{-(\alpha-\beta)^*\hat{a}} |\gamma\rangle = e^{-\frac{1}{2}|\alpha-\beta|^2} e^{(\alpha-\beta)\gamma^* - (\alpha-\beta)^*\gamma}$

$$\Rightarrow \boxed{\mathrm{Tr}\big(\hat{D}^\dagger(\alpha)\hat{D}(\beta)\big) = e^{-\frac{1}{2}|\alpha-\beta|^2} e^{\frac{\beta\alpha^* - \alpha^*\beta}{2}} \underbrace{\int \frac{d^2\gamma}{\pi} e^{(\alpha-\beta)\gamma^* - (\alpha-\beta)^*\gamma}}_{\delta^{(2)}(\alpha-\beta)} = \delta^{(2)}(\alpha-\beta)} \quad ✓$$

(d) • $\langle 0|\hat{D}(\alpha)|0\rangle = e^{-\frac{1}{2}|\alpha|^2}\langle 0|e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}}|0\rangle = e^{-\frac{1}{2}|\alpha|^2}$ ✓

• $\langle\alpha_1|\hat{D}(\alpha)|\alpha_2\rangle = e^{-\frac{1}{2}|\alpha|^2}\langle\alpha_1|e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}}|\alpha_1\rangle = e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\alpha_1^*-\alpha^*\alpha_2}$

Alternatively $\langle\alpha_1|\hat{D}(\alpha)|\alpha_2\rangle = \langle 0|\hat{D}(-\alpha_1)\hat{D}(\alpha)\hat{D}(\alpha_2)|0\rangle$

$= \langle 0|\hat{D}(-\alpha_1)\hat{D}(\alpha+\alpha_2)|0\rangle\, e^{i\,\mathrm{Im}(\alpha\alpha_2^*)} = \langle 0|\hat{D}(\alpha+\alpha_2-\alpha_1)|0\rangle\, e^{-i\,\mathrm{Im}(\alpha_1^*(\alpha+\alpha_2))}\, e^{i\,\mathrm{Im}(\alpha\alpha_2^*)}$

$= e^{-\frac{1}{2}|\alpha+\alpha_2-\alpha_1|^2}\, e^{i\,\mathrm{Im}(\alpha\alpha_2^*-\alpha_1\alpha^*-\alpha_1\alpha_2^*)}$ ✓

• $\langle n|\hat{D}(\alpha)|n\rangle = e^{-|\alpha|^2/2}\langle n|e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}}|n\rangle = e^{-|\alpha|^2/2}\sum_m \frac{(-1)^m(|\alpha|^2)^m}{(m!)^2}\underbrace{\langle n|(\hat{a}^\dagger)^m(\hat{a})^m|n\rangle}_{\text{only equal \# } \hat{a}^\dagger \text{ \& } \hat{a}}$

Now $\langle n|(\hat{a}^\dagger)^m\hat{a}^m|n\rangle = \|\hat{a}^m|n\rangle\|^2 = n(n-1)\cdots(n-m) = \frac{n!}{(n-m)!}$ $m\le n$ 0, otherwise

$$\Rightarrow \boxed{\langle n|\hat{D}(\alpha)|n\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{m=0}^{n}\binom{n}{m}\frac{(-1)^m}{m!}(|\alpha|^2)^m = e^{-\frac{|\alpha|^2}{2}}\mathcal{L}_n(|\alpha|^2)}$$

Laguerre polynomial.

Problem 2: Ways of calculating the Wigner function

We have defined $W(X,P) = \mathrm{Tr}\left(\hat{\rho}\,\delta(\hat{X}-X, \hat{P}-P)\right) = \int \frac{dX'dP'}{(2\pi)^2} e^{i(PX' - XP')} \mathrm{Tr}(\hat{\rho}\,\hat{D}(X',P'))$

where $\hat{D}(X',P') = e^{-i\hat{P}X' + i\hat{X}P'} \Rightarrow \mathrm{Tr}(\hat{\rho}\hat{D}(X',P')) = \mathrm{Tr}(\hat{\rho}\, e^{iP'\hat{X}} e^{-iX'\hat{P}})\, e^{-i\frac{X'P'}{2}}$

Aside: $\mathrm{Tr}(\hat{\rho}\, e^{iP'\hat{X}} e^{-iX'\hat{P}}) = \mathrm{Tr}(e^{-i\frac{X'}{2}\hat{P}} \hat{\rho}\, e^{iP'\hat{X}} e^{-i\frac{X'}{2}\hat{P}}) = \int dX'' \langle X''| e^{-i\frac{X'}{2}\hat{P}} |\psi\rangle\langle\psi| e^{iP'\hat{X}} e^{-i\frac{X'}{2}\hat{P}} |X''\rangle$

Since $\hat{P}$ is the generator of translations $e^{-iX_0\hat{P}}|X\rangle = |X + X_0\rangle$, $\langle X| e^{-iX_0\hat{P}} = \langle X - X_0|$

$\mathrm{Tr}(\hat{\rho}\, e^{iP'\hat{X}} e^{-iX'\hat{P}}) = \int dX''\, \psi(X'' - \frac{X'}{2})\, \psi^*(X'' + \frac{X'}{2})\, e^{iP'(X'' + \frac{X'}{2})}$

$\Rightarrow W(X,P) = \int \frac{dX''\,dX'\,dP'}{(2\pi)^2} \psi(X'' - \frac{X'}{2})\,\psi^*(X'' + \frac{X'}{2})\, e^{iPX'} e^{iP'(X'' + X)}$

$= \int \frac{dX''\,dX'}{2\pi} \psi^*(X'' + \frac{X'}{2})\,\psi(X'' - \frac{X'}{2})\,\delta(X'' + X)\, e^{iPX''}$ (having integrated over P')

$$\Rightarrow \boxed{W(X,P) = \int \frac{dX'}{2\pi} \psi^*(X + \frac{X'}{2})\,\psi(X - \frac{X'}{2})\, e^{-iPX'}}$$ Wigner's original form

(b) Marginals:

$\int dP\, W(X,P) = \int \frac{dX'}{2\pi} \psi^*(X + \frac{X'}{2})\,\psi(X - \frac{X'}{2}) \underbrace{\int dP\, e^{-iPX'}}_{2\pi\delta(X')} = \psi^*(X)\,\psi(X) = |\psi(X)|^2$ ✓

$\int dX\, W(X,P) = \int \frac{dX'}{2\pi} e^{-iPX'} \int dX\, \psi^*(X + \frac{X'}{2})\,\psi(X - \frac{X'}{2}) = \int \frac{dX'}{2\pi} e^{-iPX'} \underbrace{\int dX\, \psi^*(X)\,\psi(X - X')}_{\text{Convolution}}$

$= \int \frac{dX'}{2\pi} e^{-iPX'} \int dP' |\tilde{\psi}^*(P')|^2 e^{iP'X'} = \int dP'\, \delta(P - P')\, |\tilde{\psi}(P')|^2 = |\tilde{\psi}(P)|^2$ ✓

(c) Recall from lecture

$$W(\alpha) = \frac{2}{\pi}\int d^2\beta \; P(\beta)\, e^{-2|\beta-\alpha|^2} \quad \text{(Wigner function is smoothed P-function)}$$

$$\hat{\rho} = \int d^2\alpha \; P(\alpha)\,|\alpha\rangle\langle\alpha| \Rightarrow \langle -\beta|\hat{\rho}|\beta\rangle = \int d^2\alpha \; P(\alpha)\,\langle -\beta|\alpha\rangle\langle\alpha|\beta\rangle$$

$$\Rightarrow \langle -\beta|\hat{\rho}|\beta\rangle = \int d^2\alpha \; P(\alpha)\, e^{-\frac{1}{2}|\alpha-\beta|^2}\, e^{-\frac{1}{2}|\alpha+\beta|^2}\, e^{\beta\alpha^* - \beta^*\alpha}$$

$$\frac{e^{+|\beta|^2}}{\pi}\langle -\beta|\hat{\rho}|\beta\rangle = \int \frac{d^2\alpha}{\pi}\left(P(\alpha)\, e^{-|\alpha|^2}\right) e^{\beta\alpha^* - \beta^*\alpha} \quad \text{(2D Fourier transform)}$$

$$\Rightarrow P(\alpha) = e^{|\alpha|^2}\int \frac{d^2\beta}{\pi}\left(\frac{e^{|\beta|^2}}{\pi}\langle -\beta|\hat{\rho}|\beta\rangle\right) e^{\alpha\beta^* - \alpha^*\beta}$$

$$W(\alpha) = \frac{2}{\pi}\int d^2\gamma \; P(\gamma)\, e^{-2|\gamma-\alpha|^2} = \frac{2}{\pi}\int d^2\gamma \left[e^{|\gamma|^2}\int \frac{d^2\beta}{\pi^2}\, e^{|\beta|^2}\langle -\beta|\hat{\rho}|\beta\rangle\, e^{\gamma\beta^* - \gamma^*\beta}\right] e^{-2|\gamma-\alpha|^2}$$

$$= \frac{2}{\pi^2}\int \frac{d^2\beta}{\pi}\, e^{|\beta|^2}\langle -\beta|\hat{\rho}|\beta\rangle \int d^2\gamma \; e^{|\gamma|^2}\, e^{-2|\gamma-\alpha|^2}\, e^{\gamma\beta^* - \gamma^*\beta}$$

Aside: $e^{|\gamma|^2}e^{-2|\gamma-\alpha|^2} = e^{-|\gamma|^2 - 2(\gamma\alpha^* + \gamma^*\alpha) - 2|\alpha|^2} = e^{2|\alpha|^2}\, e^{-|\gamma-2\alpha|^2}$ (completing the square)

$$\Rightarrow \int d^2\gamma \; e^{-2|\gamma-\alpha|^2}\, e^{\gamma\beta^* - \gamma^*\beta} = e^{2|\alpha|^2}\int d^2\gamma \; e^{-|\gamma-2\alpha|^2}\, e^{\gamma\beta^* - \gamma^*\beta} = \pi e^{2|\alpha|^2}\, e^{-|\beta|^2}\, e^{2(\alpha\beta^* - \beta^*\alpha)}$$

$$\Rightarrow \boxed{W(\alpha) = \frac{2}{\pi^2}\, e^{2|\alpha|^2}\int d^2\beta \; \langle -\beta|\hat{\rho}|\beta\rangle\, e^{2(\alpha\beta^* - \beta^*\alpha)}}$$

Problem 3: Examples of Wigner Functions

(a) Fock State: $|n\rangle$

$$\boxed{Q(\alpha) = \frac{|\langle\alpha|n\rangle|^2}{\pi} = \frac{e^{-|\alpha|^2}(|\alpha|^2)^n}{\pi\, n!}}$$

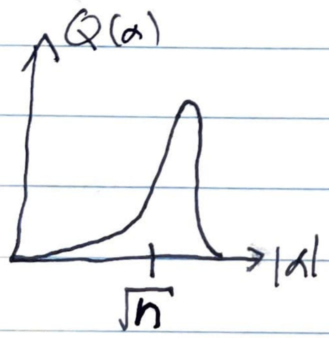
$$W(\alpha) = \int \frac{d^2\beta}{\pi}\, \chi_0(\beta)\, e^{\alpha\beta^* - \alpha^*\beta}$$

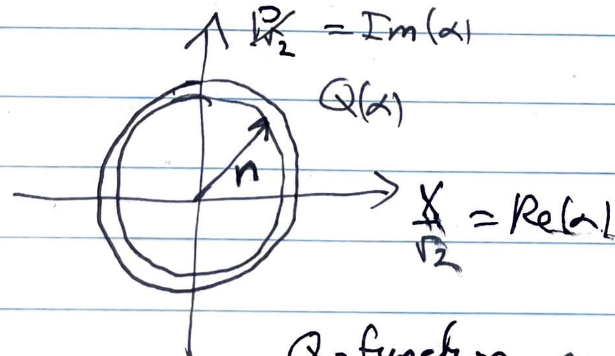
where $\chi_0(\beta) = \frac{1}{\pi}\langle n|D(\beta)|n\rangle$ is the symmetrically order characteristic function. From Problem 1

$$\chi_0(\beta) = \frac{e^{-|\beta|^2/2}}{\pi}\, \mathcal{L}_n(|\beta|^2)$$ ← Laguerre Polynomial

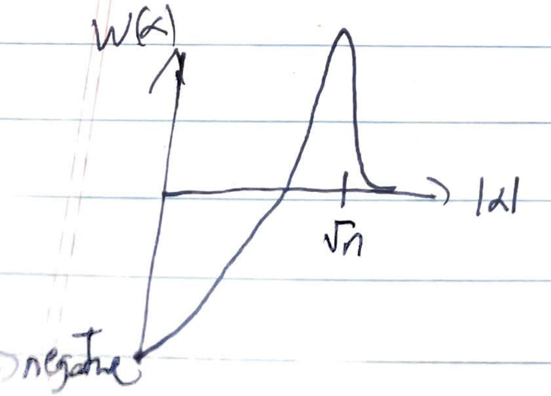
Using Mathematica:

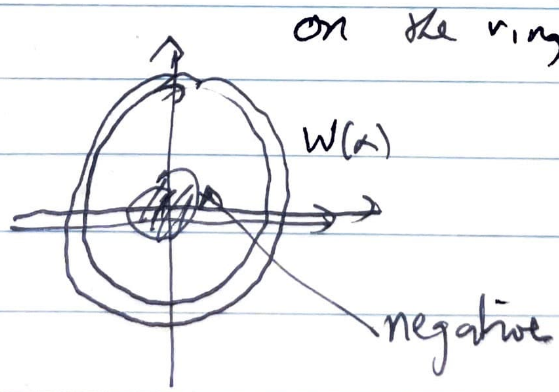
$$\boxed{W(\alpha) = \frac{2(-1)^m}{\pi}\, e^{-2|\alpha|^2}\, \mathcal{L}_n(4|\alpha|^2)}$$





Q-function concentrated on the ring





(b) "Cat State"

call this α_0

$$|\psi\rangle = \frac{1}{N}\left(|\alpha_0\rangle + e^{i\phi}|-\alpha_0\rangle\right)$$

- $$\langle\psi|\psi\rangle = \frac{1}{N^2}\Big(2 + e^{i\phi}\underbrace{\langle\alpha_0|-\alpha_0\rangle}_{e^{-2|\alpha|^2}} + e^{-i\phi}\langle-\alpha_0|\alpha_0\rangle\Big)$$

$$= \frac{2}{N^2}\left(1 + \cos\phi\, e^{-2|\alpha_0|^2}\right) = \underline{1}$$

$$\Rightarrow \boxed{N = \frac{1}{\sqrt{2}}\left(1 - \cos\phi\, e^{-2|\alpha_0|}\right)^{-1}}$$

when $|\alpha| \gg 1$

$N \approx \frac{1}{\sqrt{2}}$

- Husmi $Q(\alpha) = \frac{1}{\pi}|\langle\alpha|\psi\rangle|^2$

$$= \frac{1}{\pi N^2}\Big|\Big(\underbrace{\langle\alpha|\alpha_0\rangle}_{e^{-\frac{|\alpha-\alpha_0|^2}{2}} e^{-i\,Im(\alpha\alpha_0^*)}} + e^{i\phi}\underbrace{\langle\alpha|-\alpha_0\rangle}_{e^{-\frac{|\alpha+\alpha_0|^2}{2}} e^{i\,Im(\alpha\alpha_0^*)}}\Big)\Big|^2$$

$$\Rightarrow Q(\alpha) = \frac{1}{\pi N^2}\Big(\underbrace{e^{-|\alpha-\alpha_0|^2}}_{} + \underbrace{e^{-|\alpha+\alpha_0|^2}}_{} \underbrace{+2e^{-\frac{|\alpha-\alpha_0|}{2}} e^{-\frac{|\alpha+\alpha_0|^2}{\cancel{2}}}}_{\text{overlap}}\Big)$$

Q-function of coherent states localized at α_0 and $-\alpha_0$

Wigner function

$$W(\alpha) = \int \frac{d^2\beta}{\pi} \chi_0(\beta)\, e^{\alpha\beta^* - \alpha^*\beta}$$

$$\chi_0(\beta) = \frac{1}{\pi} \langle \psi | \hat{D}(\beta) | \psi \rangle \quad \text{(characteristic function)}$$

$$= \frac{1}{N\pi} \Big(\langle \alpha_0 | \hat{D}(\beta) | \alpha_0 \rangle + \langle -\alpha_0 | \hat{D}(\beta) | -\alpha_0 \rangle$$
$$+ e^{i\phi} \langle -\alpha_0 | \hat{D}(\beta) | \alpha_0 \rangle + e^{-i\phi} \langle \alpha_0 | \hat{D}(\beta) | -\alpha_0 \rangle \Big)$$

The first two terms represent the characteristic function of the coherent states $|\alpha_0\rangle$ and $|-\alpha_0\rangle$ respectively: $W_{\pm\alpha_0}(\alpha) = \frac{2}{\pi} e^{-2|\alpha \mp \alpha_0|^2}$

To calculate the "off-diagonal" terms, note

$$\langle -\alpha_0 | \hat{D}(\beta) | \alpha_0 \rangle = e^{-|\beta + 2\alpha_0|^2/2}$$

Need to find $\int \frac{d^2\beta}{\pi^2} e^{-\frac{|\beta + 2\alpha_0|^2}{2}} e^{\alpha\beta^* - \alpha^*\beta}$, let $\beta = \gamma - 2\alpha_0$

$$\hookrightarrow \underbrace{\int \frac{d^2\gamma}{\pi^2} e^{-\frac{|\gamma|^2}{2}} e^{\alpha\gamma^* - \alpha^*\gamma}}_{\frac{2}{\pi} e^{-2|\alpha|^2}} e^{-2\alpha_0(\alpha - \alpha^*)}$$

$$\Rightarrow \boxed{W(\alpha) = \frac{2}{N\pi} \Big\{ e^{-2|\alpha - \alpha_0|^2} + e^{-2|\alpha + \alpha_0|^2} + \underbrace{2 e^{-2|\alpha|^2} \cos(4\alpha_0 \operatorname{Im}(\alpha) + \phi)}_{\text{interference term}} \Big\}}$$

$\frac{x_0}{\sqrt{2}}$

- Take $\alpha_0 \;(= \frac{x_0}{\sqrt{2}})$ to be real, $\phi = 0$

$$\Rightarrow \boxed{W(x,p) = \frac{2}{\sqrt{\pi}^{3}}\left\{ e^{-p^2} e^{-(x-x_0)^2} + e^{-p^2} e^{-(x+x_0)^2} + 2e^{-(x^2+p^2)} \cos(2x_0 p) \right\}}$$

$$\int_{-\infty}^{\infty} dp \; W(x,p) = \frac{1}{N^2}\left(\sqrt{\frac{2}{\pi}}\, e^{-(x-x_0)^2} + \sqrt{\frac{2}{\pi}}\, e^{-(x+x_0)^2} \right)$$

$$= |\langle x|\psi\rangle|^2$$

Superposition of two Gaussian wavepackets ✓

$$\int_{-\infty}^{\infty} dx \; W(x,p) = \frac{1}{N^2}\left\{ \sqrt{\frac{2}{\pi}}\, e^{-p^2} + \sqrt{\frac{2}{\pi}}\, e^{-p^2} + \sqrt{\frac{2}{\pi}}\, 2\cos(2x_0 p) \right\}$$

$$= \frac{2}{N^2}\left(1 + \cos(2x_0 p) \right) \sqrt{\frac{2}{\pi}}\, e^{-p^2}$$

$$= \underbrace{\frac{4}{N^2} \sqrt{\frac{2}{\pi}}\, e^{-p^2} \cos^2(4x_0 p)}_{\text{Like double slit interference pattern}} = |\langle p|\psi\rangle|^2$$

Problem 4: Nonclassical States via Kerr effect

Kerr Hamiltonian $\hat{H} = \frac{\hbar K}{2} \hat{a}^{\dagger 2} \hat{a}^2 = \frac{\hbar K}{2} (\hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} - \hat{a}^\dagger \hat{a})$

$= \frac{\hbar K}{2} (\hat{n}^2 - \hat{n}) = \frac{\hbar K}{2} \hat{n}(\hat{n}-1)$

(a) At $t=0$ $|\psi(0)\rangle = |\alpha_0\rangle = \sum_n c_n |n\rangle$

$c_n = e^{-|\alpha_0|^2/2} \frac{\alpha_0^n}{\sqrt{n!}}$

$\Rightarrow |\psi(t)\rangle = e^{-\frac{i}{\hbar}Ht} |\psi(0)\rangle = \sum_n e^{-i\frac{Kt}{2} n(n-1)} c_n |n\rangle$

at $t = \frac{\pi}{K}$ $|\psi(t=\frac{\pi}{K})\rangle = \sum_n (-i)^{n^2} (+i)^n c_n |n\rangle$

Aside: $(-i)^{n^2} = (1, -i, 1, -i \ldots)$
$n = 0, 1, 2, 3$

$= \frac{1}{\sqrt{2}} \left(e^{-i\frac{\pi}{4}} + (-1)^n e^{i\pi/4} \right)$

$\Rightarrow |\psi(t=\frac{\pi}{K})\rangle = \frac{e^{-i\pi/4}}{\sqrt{2}} \sum_n i^n c_n |n\rangle + \frac{e^{i\pi/4}}{\sqrt{2}} \sum_n (-i)^n c_n |n\rangle$

$$\boxed{\Rightarrow |\psi(t=\frac{\pi}{K})\rangle = \frac{e^{-i\pi/4}}{\sqrt{2}} |i\alpha_0\rangle + \frac{e^{i\pi/4}}{\sqrt{2}} |-i\alpha_0\rangle = \frac{e^{-i\pi/4}}{\sqrt{2}} \left(|i\alpha_0\rangle + i|-i\alpha_0\rangle \right)}$$

This a "cat state"!

(b) The density operator at time t

$$\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)|$$

The Wigner function

$$W(\alpha,t) = \frac{2}{\pi^2} e^{2|\alpha|^2} \int d^2\beta \, \langle -\beta|\psi(t)\rangle\langle\psi(t)|\beta\rangle \, e^{-2(\beta\alpha^* - \beta^*\alpha)}$$

Aside: $\langle -\beta|\psi(t)\rangle = \sum_n c_n \langle -\beta|n\rangle \, e^{-i\frac{\kappa t}{2} n(n-1)}$

$$= e^{-\frac{|\alpha_0|^2}{2} - \frac{|\beta|^2}{2}} \sum_n \frac{(\alpha_0)^n (-\beta^*)^n}{n!} e^{-i\frac{\kappa t}{2} n(n-1)}$$

$$\langle\psi(t)|\beta\rangle = e^{-\frac{|\alpha_0|^2}{2} - \frac{|\beta|^2}{2}} \sum_m \frac{(\alpha_0^*)^m (\beta)^m}{m!} e^{-i\frac{\kappa t}{2} m(m-1)}$$

We must thus consider the integral

$$\int d^2\beta \, (-\beta^*)^n (\beta)^m \, e^{-2(\beta\alpha^* - \beta^*\alpha)} \, e^{-|\beta|^2/2}$$

$$= \left(\frac{1}{-2}\frac{\partial}{\partial\alpha}\right)^n \left(+\frac{1}{2}\frac{\partial}{\partial\alpha^*}\right)^m \underbrace{\int d^2\beta \, e^{-2(\beta\alpha^* - \beta^*\alpha)} \, e^{-|\beta|^2/2}}_{e^{-4|\alpha|^2}} \quad \text{(Fourier transform)}$$

$$\Rightarrow \boxed{W(\alpha,t) = \frac{2}{\pi} e^{2|\alpha|^2} e^{-|\alpha_0|^2} \sum_{n,m} \frac{1}{(-2)^{n+m}} \frac{(\alpha_0)^n}{n!} \frac{(\alpha_0^*)^m}{m!} \times e^{-i\frac{\kappa t}{2}[n(n-1) - m(m-1)]} \, \partial_\alpha^n \partial_{\alpha^*}^m \, e^{-4|\alpha|^2}}$$

One useful form

(d) The Wigner function evolves according to

$$\frac{\partial W}{\partial t} = \{H_W(\alpha,\alpha^*), W(\alpha,\alpha^*)\}_{MB} \leftarrow \text{Moyal bracket}$$

$$= H_W(\alpha,\alpha^*) \star W(\alpha,\alpha^*) - W(\alpha,\alpha^*) \star H_W(\alpha,\alpha^*)$$

star product.

We need to find the Weyl symbol for the Hamiltonian. For that purpose, we will use the Bopp representation

$$H_W(\alpha,\alpha^*) = H\left(\hat{a} = \alpha + \tfrac{1}{2}\partial_{\alpha^*},\ \hat{a}^\dagger = \alpha^* - \tfrac{1}{2}\partial_\alpha\right) 1$$

$$= \frac{\hbar\kappa}{2}\left(\alpha^* - \tfrac{1}{2}\partial_\alpha\right)^2\left(\alpha + \tfrac{1}{2}\partial_{\alpha^*}\right)^2 1 = \frac{\hbar\kappa}{2}\left(\alpha^{*2} - \alpha^*\partial_\alpha + \tfrac{1}{4}\partial_\alpha^2\right)\alpha^2$$

$$= \frac{\hbar\kappa}{2}\left(\alpha^{*2}\alpha^2 - 2\alpha\alpha^* + \tfrac{1}{2}\right) = \frac{\hbar\kappa}{2}\left(|\alpha|^2(|\alpha|^2 - 2)\right) + \frac{\hbar\kappa}{4}$$

From lecture notes:

$$\frac{\partial W}{\partial t} = \{H, W\}_{MB} = -i\left(\frac{\partial H}{\partial \alpha}\frac{\partial W}{\partial \alpha^*} - \frac{\partial H}{\partial \alpha^*}\frac{\partial W}{\partial \alpha}\right)$$

$$-\frac{i}{8}\left(\frac{\partial^3 H}{\partial^2\alpha^*\partial\alpha}\frac{\partial^3 W}{\partial^2\alpha\partial\alpha^*} - \frac{\partial^3 H}{\partial^2\alpha\partial\alpha^*}\frac{\partial^3 W}{\partial^2\alpha^*\partial\alpha}\right) + \cancel{\cdots}^{\,0}$$

This series truncates because H is 4^{th} order

After ~~algebra~~ algebra

$$\boxed{\frac{\partial W}{\partial t} = -i\kappa\left(|\alpha|^2 - 1\right)\left(\alpha^*\frac{\partial W}{\partial \alpha^*} - \alpha\frac{\partial W}{\partial \alpha}\right) - \frac{i\kappa}{4}\left(\alpha^*\frac{\partial}{\partial \alpha} - \alpha\frac{\partial}{\partial \alpha^*}\right)\frac{\partial^2 W}{\partial\alpha\partial\alpha^*}}$$

In the TWA we retain only the first term (generated by the Poisson bracket)

TWA: $$\frac{\partial W}{\partial t} = -i\kappa(|\alpha|^2-1)\left(\alpha^* \frac{\partial W}{\partial \alpha^*} - \alpha \frac{\partial W}{\partial \alpha}\right)$$

We recall from lecture $+i\left(\alpha^* \frac{\partial}{\partial \alpha^*} - \alpha \frac{\partial}{\partial \alpha}\right) = \frac{\partial}{\partial \phi}$ in polar coordinates, i.e. the generator on rotations in phase space. With $\alpha = re^{i\phi}$

TWA $$\boxed{\frac{\partial W}{\partial t} = -\kappa(r^2-1)\frac{\partial}{\partial \phi}}$$

This says the the point in phase by an amount proportional to (r^2-1). The "-1" is quantum. A flow proportional to r^2 is the phase accummulated portional to $|\alpha|^2$ — this is the Kerr effect.