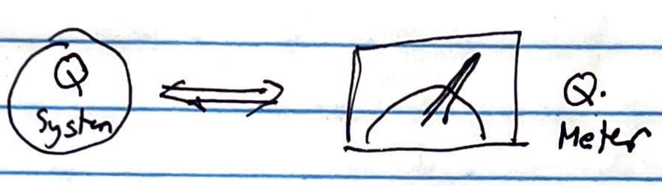
Physics 581: Open Quantum Systems

Problem Set #3: Solutions

Problem 1: QND Measurement



(a) The system and meter are coupled through the entangling unitary

$$|\Psi^{out}\rangle_{SM} = e^{-i\chi \hat{A}\otimes\hat{P}} |\psi^{in}\rangle_S \otimes |\Phi_0\rangle_M$$

Fiducial state of meter

$$|\psi^{in}\rangle_S = \sum_a c_a |a\rangle_S$$

(superposition of eigenstates of $\hat{A}$)

$$|\Psi^{out}\rangle_{SM} = \sum_a c_a e^{-i\chi\hat{A}\otimes\hat{P}} |a\rangle_S \otimes |\Phi_0\rangle_M$$

$$= \sum_a c_a |a\rangle_S \otimes \underbrace{e^{-i\chi a\hat{P}} |\Phi_0\rangle_M}_{|\Phi_{\chi a}\rangle_M} \quad (\text{since } \hat{A}|a\rangle = a|a\rangle)$$

meter displaced by an amount proportional to the eigenvalue a to be measured.

(b) Suppose we initialize the meter's pointer state in a Gaussian wave packet centered at the ~~origin~~ origin

$$\langle x|\Phi_0\rangle = \frac{1}{(2\pi\sigma_0^2)^{1/4}} e^{\frac{-x^2}{4\sigma_0^2}} \quad \text{(variance } \sigma_0^2\text{)}$$

If we measure the position of the meter, because the system and meter are entangled, the backaction on the system is determined by the measurement (Kraus) operator

$$|\tilde{\psi}_{out}\rangle_S = \hat{M}_x |\psi^{in}\rangle_S = {}_M\langle x|\Psi^{out}\rangle_{SM}$$

unnormalized state of system

$$= {}_M\langle x| e^{-i\chi\hat{A}\otimes\hat{P}} |\psi^{in}\rangle_S \otimes |\Phi_0\rangle_M$$

Aside $e^{-ix_0\hat{P}}|x\rangle_M = |x+x_0\rangle_M$ ($\hat{P}$ generates displacement)

$${}_M\langle x| e^{-ix_0\hat{P}} = {}_M\langle x-x_0|$$

$\Rightarrow$ We can write ${}_M\langle x| e^{-i\chi\hat{A}\otimes\hat{P}} = {}_M\langle x-\chi\hat{A}|$

This is crazy notation, but what is means is that x is displaced by $\chi\hat{A}$ depending on the value of $\hat{A}$ of the systems.

$$\Rightarrow |\tilde{\psi}^{out}\rangle_S = {}_M\langle x-\chi\hat{A}|\Phi_0\rangle_M |\psi^{in}\rangle_S$$

$$\Rightarrow \boxed{|\tilde{\psi}^{out}\rangle_s = \underbrace{\frac{1}{(2\pi\sigma_0^2)^{1/4}} e^{-\frac{(X-\chi\hat{A})^2}{4\sigma_0^2}}}_{\hat{M}_X \text{ (Kraus operator}} |\psi^{in}\rangle_s}$$

(c) Let's change variables and define $\mathcal{A} \equiv X/\chi$
(position of pointer in units of $\hat{A}$)

$$\hat{M}_{\mathcal{A}} \equiv \frac{1}{(2\pi\sigma_a^2)^{1/4}} e^{-\frac{(\hat{A}-\mathcal{A})^2}{4\sigma_a^2}}$$

where $\sigma_a \equiv \sigma_0/\chi$

Then $\{\hat{E}_{\mathcal{A}}\}$ forms of POVM $\quad \hat{E}_{\mathcal{A}} \equiv \hat{M}_{\mathcal{A}}^\dagger \hat{M}_{\mathcal{A}}$

$$\int d\mathcal{A}\, \hat{E}_{\mathcal{A}} = \int d\mathcal{A} \frac{1}{(2\pi\sigma_a^2)^{1/2}} e^{-\frac{(\hat{A}-\mathcal{A})^2}{2\sigma_a^2}}$$

$$= \int d\mathcal{A} \frac{1}{(2\pi\sigma_a^2)^{1/2}} \sum_a e^{-\frac{(a-\mathcal{A})^2}{2\sigma_a^2}} |a\rangle\langle a|$$

$$= \sum_a \Big(\underbrace{\int d\mathcal{A} \frac{1}{(2\pi\sigma_a^2)^{1/2}} e^{-\frac{(\mathcal{A}-a)^2}{2\sigma_a^2}}}_{1 \text{ (normalized Gaussian)}} \Big) |a\rangle\langle a|$$

$$= \sum_a |a\rangle\langle a| = \hat{1} \checkmark$$

(d) In the limit $\sigma_a \to 0$

$$\frac{1}{(2\pi\sigma_a^2)^{1/2}} e^{\frac{-(A-a)^2}{2\sigma_a^2}} \to \delta(A-a) \quad \left(\begin{matrix}\text{Delta}\\ \text{function}\end{matrix}\right)$$

$$\Rightarrow \hat{E}_A = \sum_a \frac{1}{(2\pi\sigma_a^2)^{1/2}} e^{\frac{-(A-a)^2}{2\sigma_a^2}} |a\rangle\langle a|$$

$$\Rightarrow \sum_a \delta(A-a)\,|a\rangle\langle a| = |a=A\rangle\langle a=A|$$

$\Rightarrow$ $\underline{\hat{E}_A \text{ is a projective measurement}}$

Generally the "quantum noise" in the meter's pointer (here σ_a) determines the resolution with which we can measure $\hat{A}$. The general measurement is a nonprojective POVM. We $\sigma_a \to 0$ there is no noise in the meter, and the measurement is projective.

(e) Now suppose the initial state of the system is a ~~Gauss~~ Gaussian distribution of amplitudes

$$|\langle a|\psi^{in}\rangle_s|^2 = |c^{in}(a)|^2 = \frac{1}{(2\pi\Delta A_{in}^2)^{1/2}} e^{\frac{-(a-\langle\hat{A}\rangle)^2}{2\Delta A_{in}^2}}$$

$$\Rightarrow c^{in}(a) = e^{i\phi} \frac{1}{(2\pi\Delta A_{in}^2)^{1/4}} e^{\frac{-(a-\langle\hat{A}\rangle)^2}{4\Delta A_{in}^2}}$$

unknown phase

Using quantum Bayes rule:

$$|\tilde{\psi}^{out}\rangle_S = \hat{M}_A |\psi^{in}\rangle_S$$

$$= \frac{1}{(2\pi\sigma_a^2)^{1/4}} e^{-\frac{(\hat{A}-\mathcal{A})^2}{4\sigma_a^2}} \sum_a c^{in}(a)\,|a\rangle$$

$$= \sum_a \underbrace{\frac{1}{(2\pi\sigma_a^2)^{1/4}} \frac{1}{(2\pi\Delta A^2)^{1/4}}\, e^{i\phi}\, e^{-\frac{(a-\mathcal{A})^2}{4\sigma_a^2}} e^{-\frac{(a-\langle\hat{A}\rangle)^2}{4\Delta A_{in}^2}}}_{\tilde{c}^{out}(a)} |a\rangle$$

$$\Rightarrow \tilde{P}^{out}(a) = |\tilde{c}^{out}(a)|^2 = \underset{\text{constant}}{K}\, e^{-\frac{(a-\mathcal{A})^2}{2\sigma_a^2}} e^{-\frac{(a-\langle\hat{A}\rangle)^2}{2\Delta A_{in}^2}}$$

Now we can combine the quadratic terms in the exponent to find that the output probability distribution is also Gaussian

Aside $\frac{1}{\sigma_a^2}(a-\mathcal{A})^2 + \frac{1}{\Delta A_{in}^2}(a-\langle\hat{A}\rangle)^2$

$$= \left(\frac{1}{\sigma_a^2}+\frac{1}{\Delta A_{in}^2}\right)a^2 - 2\left(\frac{\mathcal{A}}{\sigma_a^2}+\frac{\langle\hat{A}\rangle}{\Delta A_{in}^2}\right)a + \overset{\text{constant}}{C}$$

$$= \left(\frac{\sigma_a^2+\Delta A_{in}^2}{\sigma_a^2\Delta A_{in}^2}\right)\left(a-\langle\hat{A}\rangle_{out}\right)^2 + C'$$

(completing the square)

where $\langle\hat{A}\rangle_{out} = \frac{\Delta A_{in}^2}{\sigma_a^2+\Delta A_{in}^2}\mathcal{A} + \frac{\sigma_a^2}{\sigma_a^2+\Delta A_{in}^2}\langle\hat{A}\rangle_{in}$

Note: $\frac{\sigma_a^2}{\sigma_a^2 + \Delta A_{in}^2} = 1 - \frac{\Delta A_{in}^2}{\sigma_a^2 + \Delta A_{in}^2}$

$$\Rightarrow \langle \hat{A}_{out} \rangle = \langle \hat{A}_{in} \rangle + \frac{\Delta A_{in}^2}{\sigma_a^2 + \Delta A_{in}^2} (A - \langle \hat{A}_{in} \rangle)$$

Thus $\tilde{P}^{out}_{(a)} = K' e^{-\frac{1}{2\Delta A_{out}^2}(a - \langle \hat{A}_{out} \rangle)^2}$

where $\Delta A_{out}^2 = \left(\frac{\sigma_a^2}{\sigma_a^2 + \Delta A_{in}^2} \right) \Delta A_{in}^2$

Renormalizing: $K' = \frac{1}{(2\pi \Delta A_{out}^2)^{1/4}}$

The output distribution is Gaussian with a reduced "squeezed" uncertain

$$\frac{\Delta A_{out}^2}{\Delta A_{in}^2} = \frac{\sigma_a^2}{\sigma_a^2 + \Delta A_{in}^2} = \frac{1}{1 + \frac{\Delta A_{in}^2}{\sigma_a^2}}$$

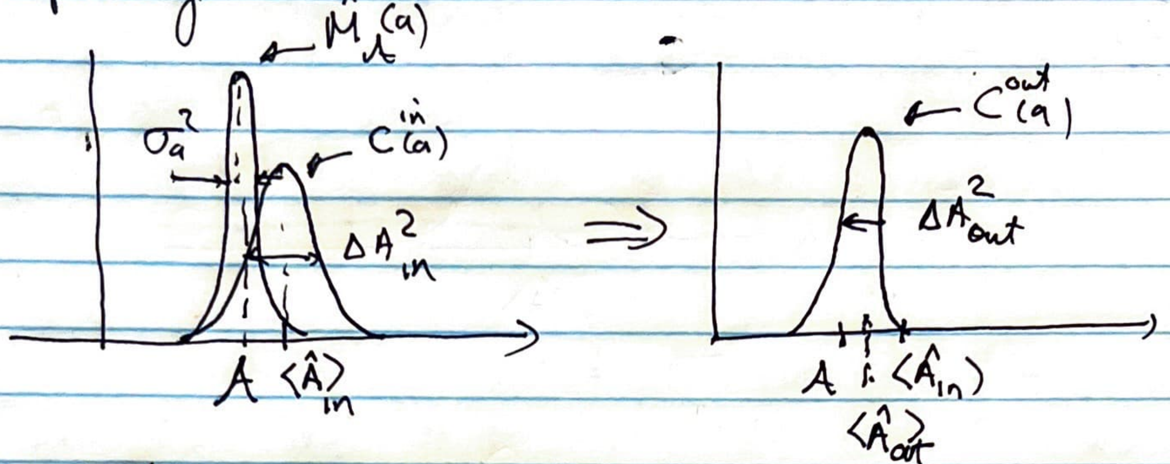
When $\sigma_a^2 \ll \Delta A_{in}^2$: $\Delta A_{out} = \sigma_a^2$,

the output distribution is squeezed by an amount depending on the pointer noise.

As $\sigma_a^2 \to 0$ $\Delta A_{out}^2 \to 0 \to$ projective

Also, $\langle \hat{A}_{out} \rangle \to \mathcal{A}$ as $\sigma_a \to 0$.

Graphically:



(f) The update we obtain for the state is according to quantum Bayes rule. The resulting ~~the~~ update on the probability distribution for the observable $\hat{A}$ measured in the QND fashion is exactly as given in classical Bayes rule.

$$p^{out}(a|\mathcal{A}) = \frac{P(\mathcal{A}|a)\, p^{in}(a)}{\| P(\mathcal{A}|a)\, p^{in}(a) \|}$$

where $\| P(\mathcal{A}|a)\, p^{in}(a) \| = P(\mathcal{A})$

Here $P(\mathcal{A}|a) = \langle a | \hat{E}_{\mathcal{A}} | a \rangle = \frac{1}{(2\pi\sigma_a^2)} e^{-\frac{(a-\mathcal{A})^2}{2\sigma_a^2}}$

= Conditional probability of finding $\mathcal{A}$ give the state was $|a\rangle$

$$p^{in}_{(a)} = |\langle a|\psi^{in}\rangle|^2 = \frac{1}{(2\pi\Delta A_{in}^2)^{1/4}} e^{\frac{-(a-\langle\hat{A}^{in}\rangle)^2}{2\Delta A_{in}^2}}$$

= Prior probability the value was "a"

Multiplying these together and renormalizing is exactly as before in the quantum case.

(g) What is quantum about quantum Bayes rule? We have seen that given a prior distribution $p^{in}(a)$ and the conditional probability of measuring A given a, we update our posterior distribut $p^{out}(a|A)$. Where is the "quantum" in quantum backaction.

The answer is that in learing $\hat{A}$ we must disturb our knowledge of any observable $\hat{B}$ that doesn't commuted with $\hat{A}$!

The backaction is on the other measurements we could have done. Thus, if we learn about, e.g., a component of angular momentum $\hat{J}_x$ and squeeze its uncertainty, we

must increase our uncertainty in $\hat{J}_y$

consistent with the uncertainty relation.

Problem 2 Quantum Channels for Qubits

(a) Given $\hat{\rho}^{in} = \begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{00} \end{bmatrix}$ we seek $\hat{\rho}^{out} = \begin{bmatrix} \rho'_{00} & \rho'_{01} \\ \rho'_{10} & \rho'_{11} \end{bmatrix} = \mathcal{E}[\hat{\rho}^{in}]$

- Depolarizing Channel $\mathcal{E}_{depol}[\hat{\rho}] = (1-p)\hat{\rho} + \frac{p}{3}\sum_{i=1}^{3} \hat{\sigma}_i \hat{\rho} \hat{\sigma}_i$

Performing matrix multiplication: $\mathcal{E}_{depol}[\hat{\rho}] = \begin{bmatrix} \rho_{00} - \frac{2}{3}p(\rho_{00}-\rho_{11}) & \rho_{01}(1-\frac{4}{3}p) \\ \rho_{10}(1-\frac{4}{3}p) & \rho_{11} - \frac{2}{3}p(\rho_{11}-\rho_{00}) \end{bmatrix}$

For a normalized state $\rho_{00} + \rho_{11} = 1 \Rightarrow \rho_{11} = 1-\rho_{00}$, $\rho_{00} = 1-\rho_{11}$

$$\Rightarrow \mathcal{E}_{depol}[\hat{\rho}] = \begin{bmatrix} \rho_{00}(1-\frac{4}{3}p) - \frac{2}{3}p & \rho_{01}(1-\frac{4}{3}p) \\ \rho_{10}(1-\frac{4}{3}p) & \rho_{11}(1-\frac{4}{3}p) - \frac{2}{3}p \end{bmatrix}$$

- Dephasing channel $\mathcal{E}_{dep}(\hat{\rho}) = (1-\frac{1}{2}p)\hat{\rho} + \frac{p}{2}\hat{\sigma}_z\hat{\rho}\hat{\sigma}_z$

Performing matrix multiplication: $\mathcal{E}_{deph}[\hat{\rho}] = \begin{bmatrix} \rho_{00} & (1-p)\rho_{01} \\ (1-p)\rho_{10} & \rho_{11} \end{bmatrix}$ (coherences are damped)

- Amplitude damping Kraus operators: $\hat{M}_0 = \begin{bmatrix} \sqrt{1-p} & 0 \\ 0 & 1 \end{bmatrix}$, $\hat{M}_1 = \begin{bmatrix} 0 & 0 \\ \sqrt{p} & 0 \end{bmatrix}$

Performing matrix multiplication: $\mathcal{E}_{amp}[\hat{\rho}] = \begin{bmatrix} (1-p)\rho_{00} & \sqrt{1-p}\,\rho_{10} \\ \sqrt{1-p}\,\rho_{01} & p\rho_{00} + \rho_{11} \end{bmatrix}$ Amplitude decay "T_1" Coherence decay "T_2"

(b+c) Heisenberg evolution of Paulis under quantum Channels

Using $\hat{\sigma}_i\hat{\sigma}_j = i\epsilon_{ijk}\hat{\sigma}_k + \delta_{ij}\hat{\mathbb{1}}$

- Depolarizing $\mathcal{E}^{+}_{depol}[\hat{\sigma}_i] = (1-p)\hat{\sigma}_i + \frac{p}{3}\sum_{j=1}^{3}\hat{\sigma}_j\hat{\sigma}_i\hat{\sigma}_j = (1-p)\hat{\sigma}_i - \frac{p}{3}\hat{\sigma}_i$

$$\Rightarrow \boxed{\mathcal{E}^{+}_{depol}[\hat{\sigma}_i] = (1-\frac{4}{3}p)\hat{\sigma}_i}$$

W.H. $\hat{\rho} = \frac{1}{2}(\hat{\mathbb{1}} + \vec{Q}\cdot\hat{\vec{\sigma}})$, $\vec{Q} = \mathrm{Tr}(\hat{\rho}\hat{\vec{\sigma}})$

$\Rightarrow \mathcal{E}[\hat{\rho}] = \frac{1}{2}(\hat{\mathbb{1}} + \vec{Q}'\cdot\hat{\vec{\sigma}})$, $\vec{Q}' = \mathrm{Tr}(\hat{\rho}'\hat{\vec{\sigma}}) = (\vec{\sigma}|\mathcal{E}|\hat{\rho}) = (\vec{\sigma}'|\hat{\rho})$

$$\hat{\vec{\sigma}}' = \mathcal{E}^\dagger[\hat{\vec{\sigma}}] = (1-\tfrac{4}{3}p)\,\hat{\vec{\sigma}}$$

$$\Rightarrow \mathcal{E}[\hat{\rho}] = \frac{1}{2}\left(\hat{\mathbb{1}} + (1-\tfrac{4}{3}p)\vec{Q}\cdot\hat{\vec{\sigma}}\right) = (1-\tfrac{4}{3}p)\left[\frac{1}{2}(\hat{\mathbb{1}} + \vec{Q}\cdot\hat{\vec{\sigma}})\right] - \frac{4}{3}p\,\frac{\hat{\mathbb{1}}}{2}$$

$$= \begin{bmatrix} (1-\tfrac{4}{3}p)\rho_{00} - \tfrac{2}{3}p & (1-\tfrac{4}{3}p)\rho_{01} \\ (1-\tfrac{4}{3}p)\rho_{01} & (1-\tfrac{4}{3}p)\rho_{11} - \tfrac{2}{3}p \end{bmatrix} \quad \text{as before}$$

<u>Dephasing channel</u>:

$$\boxed{\begin{aligned} \mathcal{E}^\dagger[\hat{\sigma}_x] &= (1-\tfrac{p}{2})\hat{\sigma}_x - \tfrac{p}{2}\hat{\sigma}_x = (1-p)\hat{\sigma}_x \\ \mathcal{E}^\dagger[\hat{\sigma}_y] &= (1-\tfrac{p}{2})\hat{\sigma}_y - \tfrac{p}{2}\hat{\sigma}_y = (1-p)\hat{\sigma}_y \\ \mathcal{E}^\dagger[\hat{\sigma}_z] &= (1-\tfrac{p}{2})\hat{\sigma}_z + \tfrac{p}{2}\hat{\sigma}_z = \hat{\sigma}_z \end{aligned}}$$

$$\Rightarrow \mathcal{E}[\hat{\rho}] = \frac{1}{2}(\hat{\mathbb{1}} + \vec{Q}'\cdot\vec{\sigma}) = \begin{bmatrix} \rho_{00} & (1-p)\rho_{01} \\ (1-p)\rho_{10} & \rho_{11} \end{bmatrix} \quad \text{as before}$$

<u>Amplitude damping channel</u>: $\mathcal{E}^\dagger[\hat{\sigma}_x] = \begin{bmatrix} 0 & \sqrt{1-p} \\ \sqrt{1-p} & 0 \end{bmatrix} = \sqrt{1-p}\,\hat{\sigma}_x$ $\quad \mathcal{E}^\dagger[\sigma_y] = i\begin{bmatrix} 0 & -\sqrt{1-p} \\ \sqrt{1-p} & 0 \end{bmatrix} = \sqrt{1-p}\,\hat{\sigma}_y$

$$\mathcal{E}^\dagger[\hat{\sigma}_z] = \begin{bmatrix} 1-2p & 0 \\ 0 & -1 \end{bmatrix} = \hat{\sigma}_z - p(\mathbb{1} + \hat{\sigma}_z) = (1-p)\hat{\sigma}_z - p\,\hat{\mathbb{1}}$$

$$\Rightarrow Q_x' = \sqrt{1-p}\,Q_x = \sqrt{1-p}\,(\rho_{01}+\rho_{10}),\quad Q_y' = \sqrt{1-p}\,Q_y = \sqrt{1-p}\,\frac{(\rho_{10}-\rho_{01})}{i},\quad Q_z' = (1-2p)\rho_{00} - \rho_{11}$$

$$\mathcal{E}_{amp}[\hat{\rho}] = \frac{1}{2}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sqrt{1-p}\,(\rho_{01}+\rho_{10})\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \sqrt{1-p}\,\frac{(\rho_{10}-\rho_{01})}{i}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + [(1-2p)\rho_{00} - \rho_{11}]\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} \frac{1}{2}(\rho_{00}+\rho_{11} + (1-2p)\rho_{00} - \rho_{11}) & \sqrt{1-p}\,\rho_{01} \\ \sqrt{1-p}\,\rho_{10} & \frac{1}{2}(\rho_{00}+\rho_{11} - (1-2p)\rho_{00} + \rho_{11}) \end{bmatrix}$$

$$= \begin{bmatrix} (1-p)\rho_{00} & \sqrt{1-p}\,\rho_{01} \\ \sqrt{1-p}\,\rho_{10} & \rho_{11} + p\rho_{00} \end{bmatrix} \quad \text{As before}$$

(d) Lindblad master equation $\hat{\rho}(t+dt) = \hat{M}_0(dt)\,\hat{\rho}(t)\,\hat{M}_0^\dagger(dt) + \sum_\mu \hat{M}_\mu(dt)\,\hat{\rho}(t)\,\hat{M}_\mu^\dagger(dt)$

where $\hat{M}_\mu(dt) = \hat{L}_\mu\sqrt{dt}$, $\hat{L}_\mu$ = Lindblad jump operator

$$\hat{M}_0(dt) = \hat{\mathbb{1}} - \frac{i}{\hbar}\hat{H}_{eff}\,dt,\quad \hat{H}_{eff} = \hat{H} - \frac{i\hbar}{2}\sum_\mu \hat{L}_\mu^\dagger\hat{L}_\mu$$

We are considering three channels (with system Hamiltonian set to zero)

(d) Depolarizing, three Lindblad jump operators $\hat{L}_i = \sqrt{\frac{\Gamma}{3}}\,\hat{\sigma}_i$

$$\hat{M}_0 = \hat{\mathbb{1}} - \frac{\Gamma}{6}\left(\hat{\sigma}_x\hat{\sigma}_x + \hat{\sigma}_y\hat{\sigma}_y + \hat{\sigma}_z\hat{\sigma}_z\right)dt = \left(1-\frac{\Gamma}{2}dt\right)\hat{\mathbb{1}}$$

- $$\hat{\rho}(t+dt) = \left(1-\frac{\Gamma}{2}dt\right)\hat{\rho}\left(1-\frac{\Gamma}{2}dt\right) + \frac{\Gamma dt}{3}\left(\hat{\sigma}_x\hat{\rho}\hat{\sigma}_x + \hat{\sigma}_y\hat{\rho}\hat{\sigma}_y + \hat{\sigma}_z\hat{\rho}\hat{\sigma}_z\right)$$

$$= (1-\Gamma dt)\hat{\rho} + \frac{\Gamma dt}{3}\left(\hat{\sigma}_x\hat{\rho}\hat{\sigma}_x + \hat{\sigma}_y\hat{\rho}\hat{\sigma}_y + \hat{\sigma}_z\hat{\rho}\hat{\sigma}_z\right)$$

$$= (1-dp)\,\hat{\rho}(t) + \frac{dp}{3}\left(\hat{\sigma}_x\hat{\rho}\hat{\sigma}_x + \hat{\sigma}_y\hat{\rho}\hat{\sigma}_y + \hat{\sigma}_z\hat{\rho}\hat{\sigma}_z\right)$$

Differential depolarizing channel with $dp = \Gamma dt$

- Parameterize $\hat{\rho} = \frac{1}{2}(\hat{\mathbb{1}} + \vec{Q}\cdot\hat{\vec{\sigma}}) \Rightarrow \sum_{i=1}^{3}\hat{\sigma}_i\hat{\rho}\hat{\sigma}_i = \frac{1}{2}(\hat{\mathbb{1}} - \vec{Q}\cdot\hat{\vec{\sigma}})$

$$\mathcal{E}_{dt}[\hat{\rho}] = \hat{\rho}(t+dt) = (1-\Gamma dt)\hat{\rho} + \frac{\Gamma dt}{3}(\mathbb{1} - \vec{Q}\cdot\vec{\sigma}) = (1-\Gamma dt)\hat{\rho} + \frac{\Gamma dt}{3}(2\hat{\mathbb{1}} - \hat{\rho})$$

$$= \left(1-\frac{4}{3}\Gamma dt\right)\hat{\rho} + \frac{2}{3}\Gamma dt\,\hat{\mathbb{1}}$$

$\Rightarrow$ The Master equation for the depolarizing channel

$$\frac{\partial\hat{\rho}}{\partial t} = -\frac{4}{3}\Gamma\hat{\rho} + \frac{2}{3}\Gamma\hat{\mathbb{1}}$$

- General solution $$\hat{\rho}(t) = \underbrace{e^{-\frac{4}{3}\Gamma t}\hat{\rho}(0)}_{\text{decaying initial condition}} + \left(1-e^{-\frac{4}{3}\Gamma t}\right)\frac{\hat{\mathbb{1}}}{2}$$ exponential rise to the steady state solution

steady state $\frac{\partial\hat{\rho}}{\partial t} = 0 = -\frac{4}{3}\Gamma\hat{\rho}_{s.s.} + \frac{2}{3}\Gamma\hat{\mathbb{1}} \Rightarrow \hat{\rho}_{ss} = \frac{1}{2}\hat{\mathbb{1}}$ (maximally mixed state)

- Depolarizing channel $\hat{\rho}(t) = \left(1-\frac{4}{3}p(t)\right)\hat{\rho}(0) + \frac{4}{3}p(t)\frac{\hat{\mathbb{1}}}{2}$ ✓

$\Rightarrow p(t) = \frac{3}{4}\left(1 - e^{-\frac{4}{3}\Gamma t}\right)$

(e) Dephasing channel: $\hat{L} = \sqrt{\frac{\Gamma}{2}}\,\hat{\sigma}_z \qquad \hat{M}_0 = \left(1-\frac{\Gamma dt}{4}\right)\hat{\mathbb{1}} \qquad \hat{M}_1 = \sqrt{\frac{\Gamma dt}{2}}\,\hat{\sigma}_z$

$\Rightarrow \hat{\rho}(t+dt) = \left(1-\frac{\Gamma}{2}dt\right)\hat{\rho} + \frac{\Gamma dt}{2}\hat{\sigma}_z\hat{\rho}\hat{\sigma}_z = \left(1-\frac{dp}{2}\right)\hat{\rho} + \frac{dp}{2}\hat{\sigma}_z\hat{\rho}\hat{\sigma}_z$

Differential dephasing channel with $dp = \Gamma dt$

$$\Rightarrow \frac{\partial\hat{\rho}}{\partial t} = -\frac{\Gamma}{2}\hat{\rho} - \frac{\Gamma}{2}\hat{\sigma}_z\hat{\rho}\hat{\sigma}_z$$

To find the solution to this equation, let's look at the equation of motion of the Bloch vector

$$\frac{d\vec{Q}}{dt} = \mathrm{Tr}\left(\vec{\sigma}\frac{d\hat{\rho}}{\partial t}\right) = -\frac{\Gamma}{2}\vec{Q} + \frac{\Gamma}{2}\mathrm{Tr}(\hat{\sigma}_z \hat{\vec{\sigma}} \hat{\sigma}_z)$$

$$\Rightarrow \frac{dQ_x}{dt} = -\Gamma Q_x \Rightarrow Q_x(t) = e^{-\Gamma t} Q_x(0), \quad \frac{dQ_y}{dt} = -\Gamma Q_y \Rightarrow Q_y(t) = e^{-\Gamma t} Q_y(0),$$

$$\frac{dQ_z}{dt} = 0 \Rightarrow Q_z(t) = Q_z(0)$$

$$\Rightarrow \hat{\rho}(t) = \begin{bmatrix} \rho_{00}(0) & \rho_{01}(0)e^{-\Gamma t} \\ \rho_{10}(0)e^{-\Gamma t} & \rho_{11}(0) \end{bmatrix}$$ dephasing channel : $1-p(t) = e^{-\Gamma t} \Rightarrow p(t) = 1 - e^{-\Gamma t}$

steady state $\rho_{s.s.} = \begin{bmatrix} \rho_{00}(0) & 0 \\ 0 & \rho_{11}(0) \end{bmatrix}$

(f) Amplitude damping channel $\hat{L} = \sqrt{\Gamma}\hat{\sigma}_-$

$$\hat{M}_0 = \mathbb{1} - \frac{\Gamma}{2}\hat{\sigma}_+\hat{\sigma}_- dt, \qquad \hat{M}_1 = \sqrt{\Gamma dt}\,\hat{\sigma}_-$$

$$\mathcal{E}_{dt}[\hat{\rho}(t)] = \hat{\rho}(t) - \frac{\Gamma}{2}(\hat{\sigma}_+\hat{\sigma}_-\hat{\rho} + \hat{\rho}\hat{\sigma}_+\hat{\sigma}_-)dt + \Gamma dt\, \hat{\sigma}_-\hat{\rho}\hat{\sigma}_+$$

$$= \begin{bmatrix} 1-\Gamma dt & 0 \\ 0 & 1 \end{bmatrix} \hat{\rho} \begin{bmatrix} 1-\Gamma dt & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \sqrt{\Gamma dt} & 0 \end{bmatrix} \hat{\rho} \begin{bmatrix} 0 & 0 \\ \sqrt{\Gamma dt} & 0 \end{bmatrix}$$

Amplitude damping channel with $dp = \Gamma dt$

Master equation $\frac{d\hat{\rho}}{dt} = -\frac{\Gamma}{2}(\hat{\sigma}_+\hat{\sigma}_-\hat{\rho} + \hat{\rho}\hat{\sigma}_+\hat{\sigma}_-) + \Gamma\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+$

Solution from Bloch vector

$$\frac{dQ_z}{dt} = -\Gamma Q_z + \Gamma \Rightarrow Q_z(t) = e^{-\Gamma t}Q_z(0) - (1-e^{-\Gamma t})\,Q_z(0)$$

$$\frac{dQ_x}{dt} = -\frac{\Gamma}{2}Q_x \Rightarrow Q_x(t) = e^{-\frac{\Gamma t}{2}}Q_x(0), \qquad Q_y(t) = e^{-\frac{\Gamma t}{2}}Q_y(0)$$

$$\Rightarrow \hat{\rho}(t) = \begin{bmatrix} e^{-\Gamma t}\rho_{00}(0) & e^{-\frac{\Gamma t}{2}}\rho_{01}(0) \\ e^{-\frac{\Gamma t}{2}}\rho_{10}(0) & \rho_{11}(0) + (1-e^{-\Gamma t})\rho_{00}(0) \end{bmatrix}$$

Amplitude damping with $p(t) = 1 - e^{-\Gamma t}$

Steady state: $\hat{\rho}_{ss.} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ (All population in ground state)