

Physics 581: Quantum Optics II

Lecture 1: Review - Quantum Coherence

Coherence

Quantum Optics is the study of quantum coherence created by, and with electromagnetic fields. Coherence is the capability of a system to exhibit interference. But what is interfering? In classical physics, waves interfere. Waves represent propagation of excitations such as vibrations of matter or vibrations of a field. In quantum physics the probability amplitudes which determine the probabilities of measurement outcomes interfere. In quantum optics these two notions of coherence come together. The electromagnetic field has a fundamental description in terms of a quantum field, and an approximate description as a classical field, in an appropriate limit. The unique aspects of quantum coherence, distinct from classical wave interference, is a central subject of quantum optics. Nonclassical light can have properties, particularly for information processing.

The central mathematical object in quantum mechanics that describes quantum coherence is the density operator $\hat{\rho}$. The density operator is the quantum state, which determines the probabilities that we assign to measurement outcomes. For example, given measurement outcomes labeled by an orthonormal basis $\{|a\rangle\}$ corresponding to the eigenvectors of a Hermitian operator, the probability

$$P_a = \langle a | \hat{\rho} | a \rangle = \text{Tr}(\hat{\rho} \hat{\Pi}_a)$$

where $\hat{\Pi}_a = |a\rangle\langle a|$ are projectors. This is an example of the Born rule. The $\{\hat{\Pi}_a\}$ represents the set of all possible measurement outcomes $\Rightarrow \sum_a \hat{\Pi}_a = \hat{1}$. The diagonal elements of the density operator are the "populations". Given a superposition $|a\rangle = c_1 |b_1\rangle + c_2 |b_2\rangle \Rightarrow P_a = \langle a | \hat{\rho} | a \rangle = |c_1|^2 \langle b_1 | \hat{\rho} | b_1 \rangle + |c_2|^2 \langle b_2 | \hat{\rho} | b_2 \rangle + \underbrace{c_1^* c_2 \langle b_1 | \hat{\rho} | b_2 \rangle + c_2^* c_1 \langle b_2 | \hat{\rho} | b_1 \rangle}_{\text{Quantum interference}}$

The off-diagonal elements of the density operators are the "coherences". They determine the capability to observe interference between alternatives.

Waves and Particles

One of the cornerstones of quantum mechanics is wave-particle duality. Quantum optics is the subject in which this takes center stage. Light has both a corpuscular particle-like character, and a continuous field-like quality. The particles are quanta - elementary excitations - of the quantum field. Each mode of the field is a quantum simple harmonic oscillator. n elementary excitations of the mode describes n -quanta (particles). What about the wave nature? For the electromagnetic field, they get interesting, because we see the two aspects of waves coming together - the wave function as a probability amplitude and the classical wave as the propagation of electromagnetic vibrations.

The classical electric field of a wave $\vec{E}(\vec{r}, t) = \vec{E}^{(+)}(\vec{r}, t) + \vec{E}^{(-)}(\vec{r}, t)$
 $\vec{E}^{(+)}(\vec{r}, t) = \int_0^{\infty} \vec{E}(\vec{r}, \omega) e^{-i\omega t} d\omega$ is the "positive frequency component," $\vec{E}^{(-)}(\vec{r}, t) = \vec{E}^{(+)*}(\vec{r}, t)$
 $\Rightarrow \vec{E}(\vec{r}, t) = \text{Re}(\vec{E}(\vec{r}, t))$, where $\vec{E}(\vec{r}, \omega) = 2\vec{E}^{(+)}(\vec{r}, \omega) =$ "complex amplitude"

We can decompose this in terms of a normal mode expansion

$$\vec{E}^{(+)}(\vec{r}, t) = \sum_{\mathbf{k}} E_{0\mathbf{k}} \alpha_{\mathbf{k}}(t) \vec{u}_{\mathbf{k}}(\vec{r}), \text{ where } \{\vec{u}_{\mathbf{k}}(\vec{r})\} \text{ are the normal}$$

modes satisfying appropriate boundary condition and $E_{0\mathbf{k}}$ is characteristic scale. $\alpha_{\mathbf{k}}(t)$ is the dimensionless complex amplitude for the mode.

For a normal mode $\alpha_{\mathbf{k}}(t) = \alpha_{\mathbf{k}} e^{-i\omega_{\mathbf{k}} t}$, where $\alpha_{\mathbf{k}} \equiv \frac{X_{\mathbf{k}} + iP_{\mathbf{k}}}{\sqrt{2}}$ convention
 $\Rightarrow \text{Re}(\alpha_{\mathbf{k}}(t)) = \frac{1}{\sqrt{2}} (X_{\mathbf{k}} \cos \omega_{\mathbf{k}} t + P_{\mathbf{k}} \sin \omega_{\mathbf{k}} t)$
quadrature of oscillation

Quantumly $\alpha_{\mathbf{k}} \rightarrow \hat{a}_{\mathbf{k}}$, $\alpha_{\mathbf{k}}^* \rightarrow \hat{a}_{\mathbf{k}}^\dagger$ The annihilation and creation operators for photons in the mode. These satisfy (Bose) commutation relations

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = 0, \quad [\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{k}'}^\dagger] = 0, \quad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'}$$

The quadratures satisfy Canonical commutation relations $[\hat{X}_{\mathbf{k}}, \hat{P}_{\mathbf{k}'}] = i \delta_{\mathbf{k}\mathbf{k}'}$

The operators $\{\hat{a}_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}^\dagger\}$ encapsulate the essence of wave-particle duality and the quantum nature of the electromagnetic field. The "classical" $\hat{a}_{\mathbf{k}}$ is the complex amplitude of the field; the quantum $\hat{a}_{\mathbf{k}}^\dagger$ and $\hat{a}_{\mathbf{k}}$ create and annihilate photons in the mode. The

fact that $\{\hat{a}_k\}$ commute implies that photons are bosons; the states are symmetric under the exchange of particles.

The quasiclassical states of the quantum field are the Glauber coherent states. We will revisit what is "coherent" about coherent states. For a given mode, this is defined by $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, eigenstates are the annihilation operator. Thus the expected value $\langle\alpha|\hat{a}(t)|\alpha\rangle = \alpha e^{-i\omega t}$ is the classical amplitude. For a multimode $|\{\alpha_k\}\rangle = \otimes |\alpha_k\rangle$ $\langle\{\alpha_k\}|\vec{E}(\vec{r}, t)|\{\alpha_k\}\rangle = \vec{E}(\vec{r}, t)$, the classical electromagnetic field. $\langle\alpha|\hat{X}|\alpha\rangle = \sqrt{2} \text{Re}(\alpha)$, $\langle\alpha|\hat{P}|\alpha\rangle = \sqrt{2} \text{Im}(\alpha)$, the classical quadratures. The coherent is a minimum uncertainty state, with equal uncertainty in the quadratures

$$\Delta X = \Delta P = \frac{1}{\sqrt{2}}, \quad \Delta X \Delta P = \frac{1}{2} \text{ (minimum product consistent w/ Heisenberg)}$$

As the coherent state can describe a field with a well defined phase of oscillation it cannot have a well-defined number of photons according to the "number-phase uncertainty relation" $\Delta n \Delta \phi \gtrsim \frac{1}{2\sqrt{n}}$. This is only an approximate uncertainty relation as there is no Hermitian operator corresponding to the "phase operator" $\hat{\phi}$, but has a more general meaning, as we will revisit. The coherent state, thus, is a superposition of number states

$$|\alpha\rangle = \sum_{n=0}^{\infty} e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$\langle \hat{n} \rangle = |\alpha|^2, \quad \Delta n = \sqrt{\langle \hat{n} \rangle}, \quad P_n = \frac{\langle \hat{n} \rangle^n}{n!} e^{-\langle \hat{n} \rangle} \quad \text{Poisson distribution}$$

The mode function $u_k(\vec{r})$ can be considered as the "wave function" of the photon, i.e., single photon in the mode $|1_k\rangle = \hat{a}_k^\dagger |0\rangle$ has the wave function $u_k(\vec{r})$. A single photon can exist as a superposition of modes, e.g. $\sum_k c_k \hat{a}_k^\dagger |0\rangle = \sum_k c_k |1_k\rangle$. The wavefunction is $\varphi(\vec{r}) = \sum_k c_k u_k(\vec{r})$. For a coherent state with classical complex amplitude $E(\vec{r})$, each photon has the same wave function $u(\vec{r}) \propto E(\vec{r})$ (units removed and normalized). The classical complex electric field thus plays the role of the wavefunction of the photon. Every photon has some wavefunction, but we have an uncertain number of them. Classical interference thus agrees with quantum interference. The classical wave intensity $|E(\vec{r})|^2$ determines the probability to detect a photon.

Classical vs. Nonclassical Light

The nonclassical nature of the electromagnetic field is seen in its photon statistics as observed in photon counting.



A beam of light incident on a "photon counter" will yield a series of discrete "clicks" corresponding to the pulse of current initiated by the photoelectric effect. The photoelectric effect, however, does not require a quantized field to explain the phenomenon. If we assume, for a fixed intensity, that photoionization occurs randomly, and uncorrelated from all other events, we assume a Poisson process. The probability of detection n photons in a time interval T is then

$$p(n, T) = \int dI P(I) e^{-\eta IT} \frac{(\eta IT)^n}{n!} \quad (\text{Mandel's formula})$$

where we have allowed for a random intensity I with probability distribution $P(I)$, and the average number of photons detected in T for a fixed intensity is $\bar{n}(I, T) = \eta IT$. This is the semiclassical classical distribution of photon counting statistics. The minimum fluctuation is known as "shot noise" with $\Delta n = \sqrt{\bar{n}} = \sqrt{\eta IT}$ for a fixed intensity. If there are classical intensity fluctuations then there will be additional fluctuations in the number of photons detected from "shot to shot" $\Rightarrow \Delta n > \sqrt{\bar{n}}$.
(Super Poissonian)

The fully quantum theory of light contains the semiclassical description as a subclass. "Classical light" with a well-defined intensity is described by the coherent states. When there is a classical statistical uncertainty in the intensity, this is classical statistical optics. In the quantum description, this is a statistical mixture of coherent states

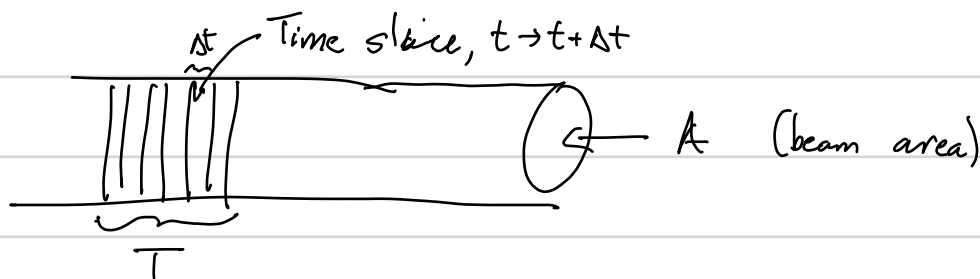
$$\hat{\rho} = \int d^2\alpha P(\alpha) |\alpha\rangle\langle\alpha|$$

This is known as the Glauber-Sudarshan P -representation of the state. When $P(\alpha) \geq 0$, this is a classical statistical mixture of coherent states, and we call

this classical light, as it reproduces the semiclassical theory. Classical light is produced by classical current $\vec{J}(\vec{r}, t)$, a C -number (stochastic) current density.

Photon Statistics and Glauber Correlation Functions

Let us consider a beam of light. For traveling waves, it is natural to consider "temporal modes"



We can consider a volume over a duration T , $V = AcT$, as a temporal mode. For a fixed intensity I , the average number of photons in the modes is $\bar{n}_T = \frac{IAT}{\hbar\omega} = |\alpha_T|^2$, where $\alpha_T = \sqrt{\frac{IAT}{\hbar\omega}} e^{i\phi}$ is the complex amplitude for some phase ϕ

We can also consider time-slice modes $t \rightarrow t + \Delta t$ $\bar{n}_{\Delta t} = \frac{IA\Delta t}{\hbar\omega} = |\alpha_{\Delta t}(t)|^2$

The coherent state factorizes $|\alpha_T\rangle = |\alpha_{\Delta t}(t_1)\rangle \otimes |\alpha_{\Delta t}(t_2)\rangle \otimes |\alpha_{\Delta t}(t_3)\rangle \otimes \dots$

There are no correlations in the counts from time-slice to time slice \Rightarrow photon counting is a Poisson process, and the counts over a given interval is a Poisson distribution.

The photon statistics within a time-slice, and correlations between time slices characterizes the nature of the light, classical vs. nonclassical.

For a coherent state with mean number \bar{n} in time-slice, $P(n) = \frac{\bar{n}^n}{n!} e^{-\bar{n}}$, Poisson distribution

A thermal state with mean number \bar{n} in time-slice, $P(n) = \frac{\bar{n}^n}{(1+\bar{n})^{n+1}}$, Bose-Einstein distribution

Note, for $\bar{n} \ll 1$ $P(n) \approx \frac{\bar{n}^n}{n!}$ (Coherent state), $P(n) = \bar{n}^n$ (Thermal state)

The thermal state's tendency to have more photons in the same mode is a result of Bose-Einstein statistics \Rightarrow Number of ways of arranging identical particles

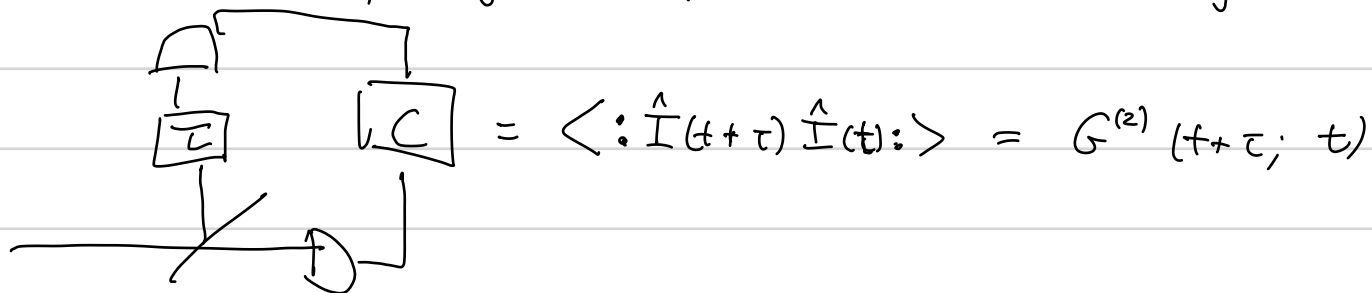
The fluctuations $\Delta n^2 = \bar{n}$ (coherent state), $\Delta n^2 = \bar{n} + \bar{n}^2$ (thermal state)

Generally, for classical light, $\Delta n^2 = \langle (a^\dagger a)^2 \rangle - \langle a^\dagger a \rangle^2 = \langle a^{\dagger 2} a^2 \rangle - \langle a^\dagger a \rangle^2 + \langle a^\dagger a \rangle$

$\Rightarrow \Delta n^2 = \langle \hat{n} \rangle + \langle \Delta k^2 \rangle$. For thermal state $P(\alpha) = \frac{1}{\pi \bar{n}} e^{-\frac{|\alpha|^2}{\bar{n}}}$ $\Rightarrow \Delta k^2 = \bar{n}^2$

\uparrow Poisson (Particle)
 \uparrow "wave" Noise

The correlation between photons in different time-slices is measured in correlation between photon counts in different detectors; e.g. two-photon correlations (Hanbury Brown Twiss)



The Glauber theory of photon counting tells us that we must look at the normally ordered operator product. In dimensionless units $\hat{I}(t) = \hat{a}^\dagger(t) \hat{a}(t)$ (we can consider this a time slice number operator, or a rate, depending on units)

$$\Rightarrow G^{(2)}(t+\tau; t) = \langle \hat{a}^\dagger(t+\tau) \hat{a}^\dagger(t+\tau) \hat{a}(t) \hat{a}(t) \rangle$$

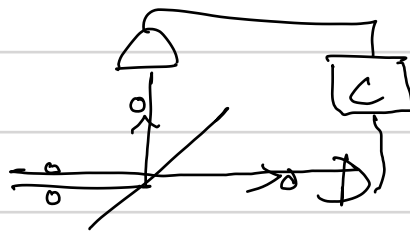
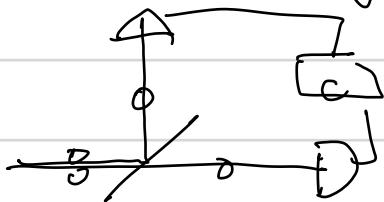
At $\tau=0$, $G^{(2)}(t) = \langle \hat{a}^\dagger(t)^2 \hat{a}(t)^2 \rangle = \#$ of coincidence counts in time slices $t \rightarrow t+\Delta t$ (note: for stationary statistics, this is independent of t , so we can call this $G^{(2)}(0) = \langle \hat{a}^\dagger{}^2 \hat{a}^2 \rangle = \langle (\hat{a}^\dagger + \hat{a})^2 \rangle - \langle \hat{a}^\dagger + \hat{a} \rangle = \langle \hat{n}(\hat{n}-1) \rangle$)

Suppose there are exactly n -photons in the time slice

↑ detecting a photon ↑ detecting another photon

$$\Rightarrow G^{(2)}(0) = n(n-1) = \frac{n!}{(n-2)!} = \binom{n}{2} 2!$$

We can interpret this as follows. The number of 2-photon coincidence counts is the number of ways of choosing two photons out of n , times the $2! = 2$ different permutations possible of the identical particles. For the HBT correlation we must add the probability amplitudes for the two indistinguishable paths



When we don't have a definite number n in the time slice

$$G^{(2)}(0) = \sum_n P(n) n(n-1) = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle$$

Consider the case $\langle \hat{n} \rangle \ll 1$ in time slice $\Rightarrow P(n) \approx \begin{cases} \langle \hat{n} \rangle^n & \text{Thermal state} \\ \frac{\langle \hat{n} \rangle^n}{n!} & \text{Coherent state} \end{cases}$

$\Rightarrow P(n) \ll 1 \quad n \gg 1$

$\Rightarrow G^{(2)}(0) \approx 2P(2) = \begin{cases} 2\langle \hat{n} \rangle^2 & \text{Thermal} \\ \langle \hat{n} \rangle^2 & \text{Coherent} \end{cases}$

For the thermal state 2x more coincidence counts for same average $\langle \hat{n} \rangle$

More generally, we define the normalized photon-photon count correlation function

$$g^{(2)}(0) = \frac{\langle : \hat{I}^2 : \rangle}{\langle \hat{I} \rangle^2} = \frac{\langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle}{\langle \hat{a}^{\dagger} \hat{a} \rangle^2} = \frac{\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle}{\langle \hat{n} \rangle^2} = \frac{\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 - \langle \hat{n} \rangle}{\langle \hat{n} \rangle^2} + 1$$

$$g^{(2)}(0) = 1 + \frac{\Delta n^2 - \langle \hat{n} \rangle}{\langle \hat{n} \rangle^2}$$

Mandel Q-factor

$g^{(2)}(0)$ measure the super vs. sub Poissonian nature of the photon number fluctuations in a temporal mode

For a coherent state $\Delta n^2 = \langle n \rangle$, $g^{(2)}(0) = 1$ Poissonian

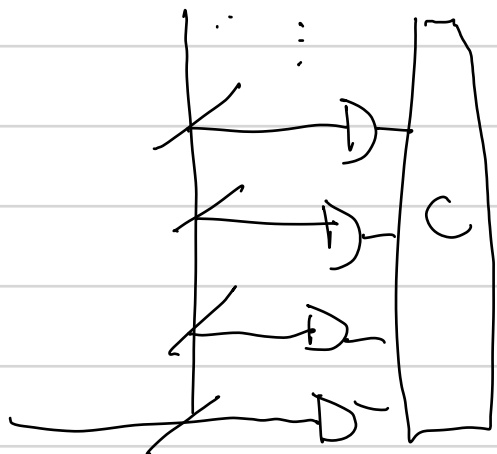
For a thermal state $\Delta n^2 = \langle \hat{n} \rangle^2 + \langle \hat{n} \rangle \Rightarrow g^{(2)}(0) = 2$ Super Poissonian

For Classical light $\Delta n^2 \geq \langle n \rangle \Rightarrow \boxed{g^{(2)}(0) < 1 \quad \text{non Classical}}$

Note: For a single photon state $\Delta n = 0$, $\langle \hat{n} \rangle = 1$, $g^{(2)}(0) = 0$ as expected

\Rightarrow No coincidence counts: Photon is indivisible

For an m-fold correlation



$$G^{(m)}(0) = \langle : \hat{I}^m : \rangle = \langle \hat{a}^{\dagger m} \hat{a}^m \rangle = \langle \hat{n}(\hat{n}-1)(\hat{n}-2) \dots (\hat{n}-m+1) \rangle$$

For a n-photon state

$$G^{(m)}(0) = \frac{n!}{(n-m)!} = \binom{n}{m} m!$$

number of ways of choosing m-photons out of n \times number of permutations = number of indistinguishable paths leading to coincidence

Again with an indefinite # of photons in the mode

$$G^{(m)}(0) = m! \sum_{n=0}^{\infty} P(n) \binom{n}{m} \approx m! P(m) \quad \text{when } P(n) \ll 1 \text{ as } n \text{ grows}$$

$$\Rightarrow G^{(0)}(0) \approx \begin{cases} 3! \langle \hat{n} \rangle^3 & \text{Thermal} \\ \langle \hat{n} \rangle^3 & \text{Poisson} \end{cases} \quad (\text{actually exact})$$

$$\text{With } g^{(m)}(0) = \frac{\langle : \hat{I}^m : \rangle}{\langle \hat{I} \rangle^m} \quad g^{(m)}(0) = \begin{cases} m! & \text{Thermal} \\ 1 & \text{Coherent} \end{cases}$$

The fact that $g^{(m)}(0) = 1 \quad \forall m \Rightarrow \langle : \hat{I}^m : \rangle = \langle \hat{I} \rangle^m \quad \forall m$ is what Glauber called the coherent state "coherent". There are no intensity fluctuations at any order \Rightarrow Perfect "coherence" of field.

Now let us consider correlations between two time slices

$$G^{(2)}(\tau) = \langle : \hat{I}(\tau) \hat{I}(0) : \rangle \leq \langle : \hat{I} : \rangle^2 = G^{(2)}(0) \quad \text{for classical light}$$

Photon antibunching

Consider the two-time intensity correlation function

$$G^{(2)}(\tau) = \langle : \hat{I}(0) \hat{I}(\tau) : \rangle = \int d[\mathcal{E}] P[\mathcal{E}] |\mathcal{E}(0)|^2 |\mathcal{E}(\tau)|^2 = \langle |\mathcal{E}(0)|^2 | \mathcal{E}(\tau) |^2 \rangle \quad (\text{inner product of function})$$

$$\text{Cauchy-Schwartz inequality: } \langle |\mathcal{E}(0)|^2 | \mathcal{E}(\tau) |^2 \rangle \leq \sqrt{\langle |\mathcal{E}(0)|^2 | \mathcal{E}(0) |^2 \rangle} \sqrt{\langle |\mathcal{E}(\tau)|^2 | \mathcal{E}(\tau) |^2 \rangle} = \sqrt{\langle : \hat{I}(0) : \rangle} \sqrt{\langle : \hat{I}(\tau) : \rangle} = \langle : \hat{I}^2(0) : \rangle$$

stationary statistics

$$\Rightarrow \langle : \hat{I}(0) \hat{I}(\tau) : \rangle = G^{(2)}(\tau) \leq \langle : \hat{I}^2(0) : \rangle = G^{(2)}(0)$$

\Rightarrow Coincident counts in same time bin \geq Coincidence counts for bins separated by time $\tau > 0$

\Rightarrow Photons tend to arrive in bunches. Classical intensity fluctuations correspond to photons bunched together. This is a consequence of Bose statistics. Bosons are more likely to be in the same (temporal) mode than distinguishable particles (or fermions).

Normalized: $g^{(2)}(0) \geq g^{(2)}(\tau)$: Photon Bunching (classical light)

Nonclassical light $g^{(2)}(0) \leq g^{(2)}(\tau)$:

\Rightarrow Rate of coincidence at zero delay \leq Rate of coincidence at finite delay.

Photons are more likely to arrive separately than together

Nonclassical light \Rightarrow Photon Antibunching

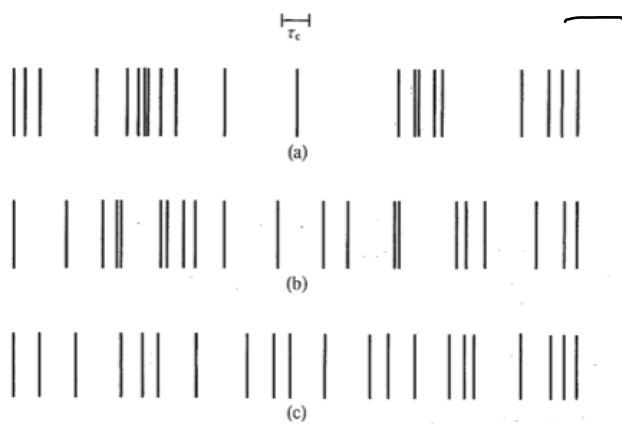
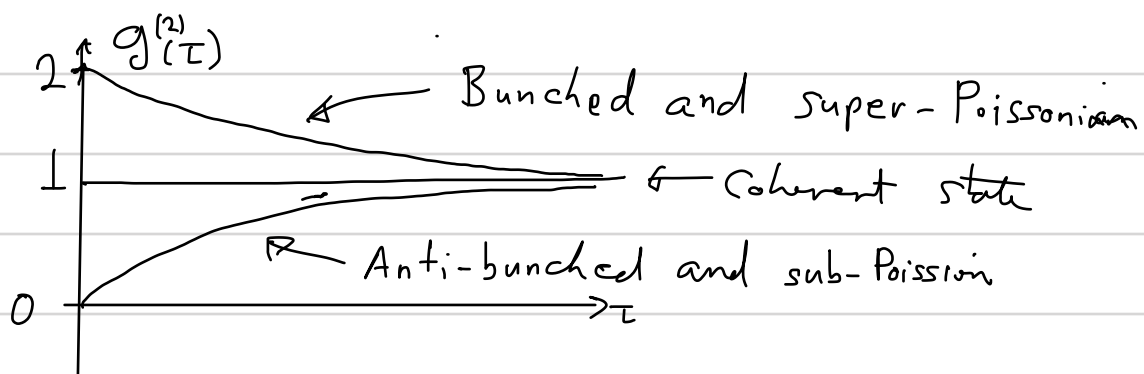


FIG. 6.4. Schematic representations of photon counts as functions of the time for light beams that are (a) bunched with $g^{(2)}(0) > 1$, (b) random with $g^{(2)}(0) = 1$, and (c) antibunched with $g^{(2)}(0) < 1$.

Taken from Loudon

"The Quantum theory of light"

The observation of photon anti-bunching and sub-Poissonian photon statistics was a major milestone in the study of nonclassical light.