Physics 581 - Quantum Optics II
Lecture 4: Quasiprobability Functions of the Field

Motivation

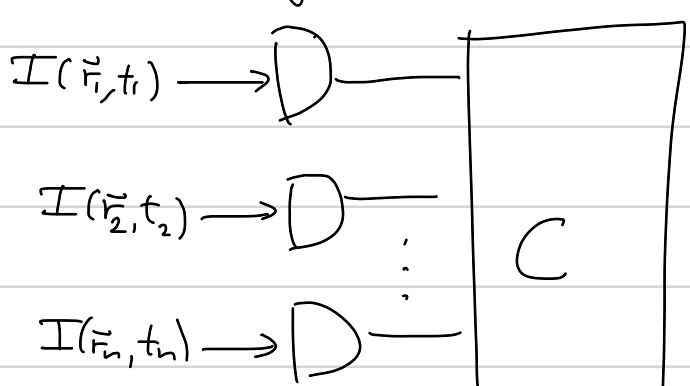
We have defined "classical light" as that whose state can be expressed as a statistical mixture of coherent state

$$\hat{\rho} = \int d^2\{\alpha_k\}\, P(\{\alpha_k\})\, |\{\alpha_k\}\rangle\langle\{\alpha_k\}|$$

$P(\{\alpha_k\})$ = Glauber-Sudharshan "P-representation". Classical $\Rightarrow$ $P(\{\alpha_k\}) \geq 0$

This definition of classicality is defined by the condition that a photon counting experiment is describable by the semiclassical theory (i.e. classical wave inducing photo-electric emission).

E.g. Coincidence counting from n-detectors



$$C = \langle :\hat{I}(\vec{r}_1,t_1)\,\hat{I}(\vec{r}_2,t_2)\cdots\hat{I}(\vec{r}_n,t_n):\rangle \qquad \hat{I}(\vec{r},t) = \hat{E}^{(-)}(\vec{r},t)\,\hat{E}^{(+)}(\vec{r},t)$$

normal order

$$= \sum(\;)\,\langle \hat{a}^\dagger_{k_1}\hat{a}^\dagger_{k_2}\cdots\hat{a}^\dagger_{k_n}\hat{a}_1\hat{a}_2\cdots\hat{a}_n\rangle$$

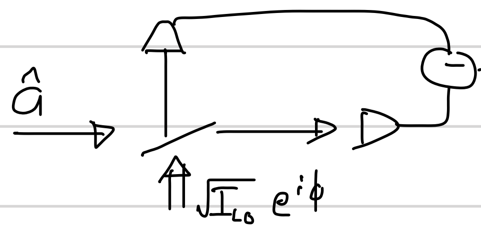
$$\hat{\rho}_{classical} \Rightarrow C = \overline{|E_1|^2\,|E_2|^2\cdots|E_n|^2}$$

Classical statical average

$$= \sum(\;)\int d^2\{\alpha_k\}\, P(\{\alpha_k\})\, |\alpha_1|^2|\alpha_2|^2\cdots|\alpha_n|^2$$

Any state of the field arising from a classical (maybe noise) current source is described by a statistical mixture of coherent states, so in this sense the light is classical.

We can also, however, consider other detection methods, and ask whether there is a classical, statistical description of the process. Consider, e.g., homodyne detection:



$$\hat{I}_{out} \propto \hat{X}_\phi = \frac{e^{-i\phi}\hat{a} + e^{i\phi}\hat{a}^\dagger}{\sqrt{2}}$$

$$\langle \hat{X}^2 \rangle = \left\langle \left(\frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}}\right)^2 \right\rangle = \frac{1}{2}\langle \underbrace{\hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger}_{\text{symmetric order}} \rangle$$

A squeezed state is non-classical in the sense that it cannot be represented as a statistical mixture of coherent states (and thus does not arise as the radiation of a classical noise source), but note

$$\langle \hat{X}^2 \rangle = \int dX \, \frac{e^{-\frac{X^2}{2\Delta X^2}}}{\sqrt{2\pi\Delta X^2}} \, X^2 = \text{classical statistical}$$

average, even when $\Delta X^2 < \frac{1}{2}$ (vacuum limit). So in some sense, the squeezed vacuum can be understood in a classical statistical noise theory.

That is, there exists another way of representing the state in terms of coherent states, such that

$$\langle \hat{X}^2 \rangle = \left\langle \left(\frac{\hat{a}+\hat{a}^\dagger}{2}\right)^2 \right\rangle = \int d^2\alpha \, W(\alpha) \left(\frac{\alpha+\alpha^*}{\sqrt{2}}\right)^2$$

$W(\alpha) \equiv$ "Wigner function"

Quantum Mechanics in Phase Space

The fact that there are many different ways to represent a state in terms coherent states follows from the fact that they are "phase space eigenstates."

$$(\hat{X}+i\hat{P})\,|\alpha\rangle = (X+iP)\,|\alpha\rangle \qquad \begin{array}{l} X=\sqrt{2}\,\mathrm{Re}(\alpha) \\ P=\sqrt{2}\,\mathrm{Im}(\alpha)\end{array}$$

$|\alpha\rangle$ is not an eigenstate of a Hermitian operator, and thus, they are not orthogonal, $|\langle\alpha|\alpha'\rangle|^2 = e^{-|\alpha-\alpha'|^2} \Rightarrow \{|\alpha\rangle\}$ is an <u>over-complete</u> basis: $\int d^2\alpha\, |\alpha\rangle\langle\alpha| = \mathbb{1}$, but smaller sets will do.

The question of how to represent quantum states in phase space has a long history. From a fundamental perspective, it allows us to compare classical and quantum behavior. Classically, we can formulate Hamiltonian dynamics via Liouville theory.

Classical State: Probability distribution in phase space $W(\vec{q},\vec{p},t)$

Liouville equation for dynamics:

$$\frac{dP}{dt} = -\underbrace{\{H(\vec{q},\vec{p}),\, W(\vec{q},\vec{p},t)\}}_{\text{Poisson bracket}} = \sum_i \left(\frac{\partial H}{\partial q_i}\frac{\partial W}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial W}{\partial q_i}\right)$$

$$\overline{f(\vec{q},\vec{p})}\Big|_t = \int d^n\vec{q}\, d^n\vec{p}\; W(\vec{q},\vec{p},t)\, f(\vec{q},\vec{p})$$

expected value of some observable function of the canonical coordinates.

As discussed in Lecture 1, classical statistical physics can be described by the Liouville equation

$$\frac{\partial}{\partial t} P(\vec{q},\vec{p},t) = \{H(\vec{q},\vec{p}),\ P(\vec{q},\vec{p},t)\} + \mathcal{L}_{diss}[P(\vec{q},\vec{p},t)]$$

Here $P(\vec{q},\vec{p},t)$ is the probability distribution on phase space which reflects our incomplete knowledge of the exact microstate specified by the point in phase space $(\vec{q},\vec{p})$. The Hamiltonian flow is governed by the Poisson bracket $\{H(\vec{q},\vec{p}), P(\vec{q},\vec{p},t)\} = \frac{\partial H}{\partial \vec{q}}\frac{\partial P}{\partial \vec{p}} - \frac{\partial H}{\partial \vec{p}}\frac{\partial P}{\partial \vec{q}}$ and the dissipative (open system evolution) is governed by the "Liouvillian" map $\mathcal{L}_{diss}[P(\vec{q},\vec{p},t)]$.

In our survey of open quantum systems, we will see the analogous Lioville description. Of particular interest is to explore the conditions under which the quantum description reduces to classical statistical mechanics. In particular, we will see how decoherence is essential in understanding the quantum-to-classical transition.

<u>Introduction to Phase Space Representations</u>

Obviously, quantum statistical physics differs from classical statistical physics; we cannot generally specify a state in terms of a joint probability on phase space. However, we can ask how close we can get, and in that way uncover the particularities that distinguish quantum and classical physics.

To begin, we know the we can specify a probality distribution in position space

$$P(x) = \langle x|\hat{\rho}|x\rangle = \mathrm{Tr}(\hat{\rho}\ |x\rangle\langle x|)$$

For opertators that are solely functions of position, e.g. potential energy $V(\hat{x})$

$$\langle V(\hat{x})\rangle = \int dx\ P(x)\, V(x) \qquad \text{where} \qquad V(x) = \langle x|V(\hat{x})|x\rangle$$

In the position representation $\hat{x} \Rightarrow x$. Let us introduce the delta function $\delta(\hat{x}-x)$. Recalling $f(\hat{A}) = \sum_a f(a)|a\rangle\langle a|$ where $\hat{A} = \sum a|a\rangle\langle a|$

$$\delta(\hat{x}-x) = \int dx'\, \delta(x'-x)\, |x'\rangle\langle x'| = |x\rangle\langle x|$$

This delta function is thus equal to the <u>projector operator</u> on $|x\rangle$

$$\Rightarrow \quad P(x) = \langle x|\hat{\rho}|x\rangle = Tr(\hat{\rho}\,\delta(\hat{x}-x))$$

Similarly, for the momentum representation

$$P(p) = \langle p|\hat{\rho}|p\rangle = Tr(\hat{\rho}\,|p\rangle\langle p|) = Tr(\hat{\rho}\,\delta(\hat{p}-p))$$

For operators that are solely a function of momentum, e.g. kinetic energy, we can calculate expectation values as stastical averages

$$\left\langle \frac{\hat{p}^2}{2m} \right\rangle = \int dp\, \frac{p^2}{2m}\, P(p)$$

But what about a joint probability distribution?
We might define $P(x,p) = Tr(\hat{\rho}\,\delta(\hat{x}-x)\,\delta(\hat{p}-p))$, but the problem is that x and p dont commute. Who is to say $\hat{x}$ comes before $\hat{p}$ and not $\hat{p}$ before $\hat{x}$. More over $[\delta(\hat{x}-x)\,\delta(\hat{p}-\hat{p})]^\dagger \neq \delta(\hat{x}-x)\,\delta(\hat{p}-p)$ so this is nothing like a probability distribution as the trace with $\hat{\rho}$ is complex.

A natural object to consider is a <u>symmetric delta function</u> $\delta(\hat{x}-x, \hat{p}-p)$. We will define what this means in a moment. Given such as object, we define

$$\boxed{W(x,p) \equiv Tr(\hat{\rho}\ \delta(\hat{x}-x, \hat{p}-p))} = \text{Wigner function}$$

The Wigner function is an example of a phase-space representation of the quantum state. However, as we will see, the operator $\delta(\hat{x}-x, \hat{p}-p)$ is not positive. The $W(x,p)$ <u>can be negative</u>. The Wigner function is not generally a probability distribution. It is a <u>quasi probability</u> distribution.

Operator Plane Wave and Delta Functions on Phase Space

In Lecture 5, we defined the two dimensional delta function on phase space

$$\delta(X-X_0, P-P_0) = \int \frac{dX'dP'}{(2\pi)^2} e^{-i(P-P_0)X' + i(X-X_0)P'} \qquad \left(\text{Fourier transform of plane wave}\right)$$

We thus define the (symmetric) delta function operator

$$\delta(\hat{X}-X_0, \hat{P}-P_0) = \int \frac{dX'dP'}{(2\pi)^2} e^{-i(\hat{P}-P_0)X' + i(\hat{X}-X_0)P'} = \int \frac{dX'dP'}{(2\pi)^2} e^{i(P_0X' - X_0P')} \underbrace{e^{-i\hat{P}X' + i\hat{X}P'}}_{\hat{D}(X',P')}$$

- Displacement operator: operator plane wave
- Symmetric delta function operator: 2D Fourier transform of Displacement operator

Note $\hat{D}(X',P') = e^{-i\hat{P}X' + i\hat{X}P'} = \sum_n \frac{(-i\hat{P}X' + i\hat{X}P')^n}{n!}$ is <u>symmetrically ordered</u> in $\hat{X}$ and $\hat{P}$.

Written in complex form: $\boxed{\delta^{(2)}(\hat{a}-\alpha, \hat{a}^\dagger-\alpha^*) = \int \frac{d^2\beta}{\pi^2} e^{\alpha\beta^* - \alpha^*\beta} \hat{D}(\beta)}$

Operator Basis

As plane waves form a complete orthonormal basis for function on phase space, the operator equivalent, the displacement operators, form a basis for operator functions of $\hat{X}$ and $\hat{P}$. Using our superator notation

$$(\hat{D}(\alpha)|\hat{D}(\beta)) = \mathrm{Tr}(\hat{D}^\dagger(\alpha)\hat{D}(\beta)) = \pi\,\delta^{(2)}(\alpha-\beta) \quad \text{(orthogonal)}$$

$$\int \frac{d^2\beta}{\pi} |\hat{D}(\beta))(\hat{D}(\beta)| = \hat{\mathbb{I}} \quad \text{(span operator space)}$$

$\hat{A} \Rightarrow |\hat{A}) = \int \frac{d^2\beta}{\pi} |\hat{D}(\beta))(\hat{D}(\beta)|\hat{A})$: $\boxed{\hat{A} = \int \frac{d^2\beta}{\pi} \mathrm{Tr}(\hat{D}^\dagger(\beta)\hat{A})\hat{D}(\beta)}$

The symmetric delta functions also form a operator basis.

Let us define $\hat{T}(\alpha) = \pi\,\delta^{(2)}(\hat{a}-a, \hat{a}^\dagger-\alpha^*) = \int \frac{d^2\beta}{\pi} e^{\alpha\beta^* - \alpha^*\beta}\hat{D}(\beta)$ (Fourier transform of Displacement)

Note $\hat{T}^\dagger(\alpha) = \hat{T}(\alpha)$

$$\hat{D}(\beta) = \int \frac{d^2\alpha}{\pi} e^{\beta\alpha^* - \beta^*\alpha}\hat{T}(\alpha) \quad \text{(inverse Fourier)}$$

$$\Rightarrow \int \frac{d^2\alpha}{\pi} |\hat{T}(\alpha)) (\hat{T}(\alpha)| = \hat{\mathbb{I}}$$

From this we come to the fundamental phase space relation.
For any operator $\hat{A}$, we define the "Weyl Symbol" $A_w(\alpha)$

$$A_w(\alpha) \equiv (\hat{T}(\alpha)|\hat{A}) = \pi \,\mathrm{Tr}(\hat{A}\, \delta(\hat{a}-\alpha, \hat{a}^\dagger-\alpha^*)) = 2\pi \,\mathrm{Tr}(\hat{A}\, \delta(\hat{x}-x, \hat{p}-p))$$

$$\langle\hat{A}\rangle = \mathrm{Tr}(\hat{\rho}\hat{A}) = (\hat{\rho}|\hat{A}) = \int \frac{d^2\alpha}{\pi} (\hat{\rho}|\hat{T}(\alpha))(\hat{T}(\alpha)|\hat{A}) = \int \frac{d^2\alpha}{\pi} \underset{\text{real}}{\rho_w(\alpha)}\, A_w(\alpha)$$

The Wigner function (with usual normalization) $W(\alpha) = \frac{1}{\pi}\rho_w(\alpha)$

$$\Rightarrow \boxed{\langle\hat{A}\rangle = \int d^2\alpha\, W(\alpha)\, A_w(\alpha) = \int dX\, dP\, W(X,P)\, A_w(X,P)}$$

We thus have a phase space representation of quantum mechanics which has great resemblance to classical statistical mechanics. Except $W(x,p)$ is generally a <u>quasi probability</u> distribution which can be negative.

<u>Properties of the Wigner Function</u>

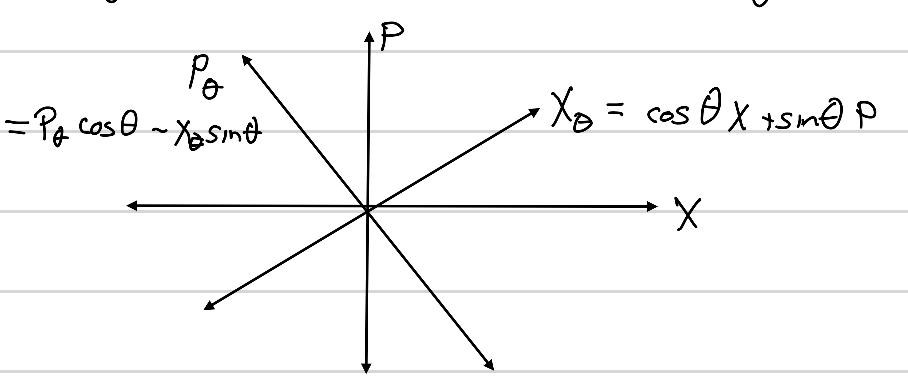
- $W(\alpha)$ is <u>real</u>.
- <u>Normalization</u> $\mathrm{Tr}(\hat{\rho})=1 \Rightarrow \int d^2\alpha\, W(\alpha) = \int dx\,dP\, W(x,P) = 1$
- <u>Marginals</u> $W(X,P)$ gives the correct marginal probability distributions

$$\int_{-\infty}^{\infty} dP\, W(x,P) = P(x) = \langle x|\hat{\rho}|x\rangle, \qquad \int_{-\infty}^{\infty} dX\, W(X,P) = P(P) = \langle P|\hat{\rho}|P\rangle$$

To see this, note $\int_{-\infty}^{\infty} dP\, \delta(\hat{x}-x, \hat{p}-P) = \delta(\hat{x}-x) = |x\rangle\langle x|$

$$\int_{-\infty}^{\infty} dX\, \delta(\hat{x}-x, \hat{p}-P) = \delta(\hat{p}-P) = |P\rangle\langle P|$$

More general the Wigner function gives the correct marginals for any quadratures



$$\int_{-\infty}^{\infty} dP_\theta\, W(X_\theta, P_\theta) = P(X_\theta) = \langle X_\theta|\hat{\rho}|X_\theta\rangle$$

$$\int_{-\infty}^{\infty} dX_\theta\, W(X_\theta, P_\theta) = P(P_\theta) = \langle P_\theta|\hat{\rho}|P_\theta\rangle$$

- Negativity: $W(x,p)$ can be negative. It is a quasiprobability distribution. "Negative probabilities" distinguish quantum from classical mechanics

- Hudson Theorem: Given a pure state $\hat{\rho} = |\psi\rangle\langle\psi|$, $W(x,p)$ is everywhere positive iff $W(x,p)$ is a Gaussian function. Gaussian (pure) states are essentially classical, in that the measurement outcomes of any quadrature X_θ have a classical statistical interpretation. Non Gaussian Wigner functions associated with pure states are a quantum "resource." Further note, the Gaussian cannot have arbitrary localization. For a physical state we must have $\Delta X \Delta P \geq \frac{1}{2}$.

- Expectation Values: $\langle\hat{A}\rangle = Tr(\hat{\rho}\hat{A}) = \int dx\, dp\, W(x,p)\, A_w(x,p)$

Where $A_w(x,p)$ is the Weyl Symbol of $\hat{A}$. Note, if $\hat{A} = f(\hat{X},\hat{P})$ is a symmetric function of $\hat{X}$ and $\hat{P}$, e.g. $f(\hat{X},\hat{P}) = \hat{X}\hat{P} + \hat{P}\hat{X}$, then $A_w(X,P) = f(X,P) = 2XP$

- Calculating the Wigner function

In complex variables, we have $W(\alpha) = \frac{1}{\pi} Tr(\hat{\rho}\, \hat{T}(\alpha)) = Tr(\hat{\rho}\, \delta(\hat{a}-\alpha, \hat{a}^\dagger - \alpha^*))$

Where $\delta(\hat{a}-\alpha, \hat{a}^\dagger - \alpha^*) = \int \frac{d^2\beta}{\pi^2} e^{\alpha\beta^* - \alpha^*\beta} \hat{D}(\beta)$

$$\Rightarrow \boxed{W(\alpha) = \int \frac{d^2\beta}{\pi} e^{\alpha\beta^* - \alpha\beta} \chi(\beta) \quad \text{where} \quad \chi(\beta) = \frac{1}{\pi} Tr(\hat{\rho}\hat{D}(\beta)) = \frac{1}{\pi}\langle\hat{D}(\beta)\rangle}$$

$\chi(\beta)$ is known as the "characteristic function"

The Wigner function is the inverse Fourier transform of the characteristic function. Thus, one way to calculate $W(\alpha)$ is first calculate $\chi(\beta)$ and then take the inverse. Conversely, the characteristic function is the fourier transform of the Wigner function.

$$\chi(\beta) = \int \frac{d^2\alpha}{\pi} e^{\beta\alpha^* - \beta^*\alpha} W(\alpha)$$

The use of the characteristic function is a standard tool in classical statistics; the Fourier transform of probability distribution is

$$\chi(k) = \int_{-\infty}^{\infty} dx \; e^{-ikx} \, P(x)$$

This is a "moment generating" function. The moments of the distribution are the derivatives of the characteristic function at the origin

$$\frac{1}{(-i)^n} \frac{d^n}{dk^n} \chi(k) \Big|_{k=0} = \int_{-\infty}^{\infty} dx \; x^n \, P(x) = \overline{x^n}$$

Similarly for the quasiprobability

$$\frac{\partial^n}{\partial \beta^n} \frac{\partial^m}{\partial(-\beta^*)} \chi(\beta, \beta^*) \Big|_{\beta=\beta^*=0} = \int d^2\alpha \, (\alpha^*)^n (\alpha)^m \, W(\alpha, \alpha^*) = \langle \{ \hat{a}^{\dagger n} \hat{a}^m \} \rangle_{sym}$$

Here we used that the Weyl symbol of symmetrically ordered $\{\hat{a}^{\dagger n} \hat{a}^m\}_{sym}$ is $\alpha^{*n} \alpha^m$

Examples:

- Wigner function of a coherent state $|\psi\rangle = |\alpha_0\rangle$

  Characteristic function: $\chi(\beta) = \frac{1}{\pi} \langle \alpha_0 | \hat{D}(\beta) | \alpha_0 \rangle = \frac{1}{\pi} \langle \alpha_0 | e^{\beta \hat{a}^\dagger - \beta^* \hat{a}} | \alpha_0 \rangle$

$$= \frac{1}{\pi} e^{-\frac{1}{2}|\beta|^2} \langle \alpha_0 | e^{\beta \hat{a}^\dagger} e^{-\beta^* \hat{a}} | \alpha_0 \rangle = \frac{1}{\pi} e^{-\frac{1}{2}|\beta|^2} e^{\alpha_0^* \beta - \alpha_0 \beta^*}$$

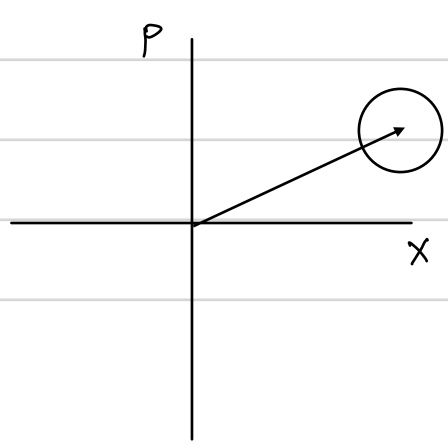
$$\Rightarrow \quad W(\alpha) = \int \frac{d\beta}{\pi} e^{\alpha \beta^* - \alpha^* \beta} \chi(\beta) = \int \frac{d^2\beta}{\pi^2} e^{(\alpha - \alpha_0)\beta^* - (\alpha - \alpha_0)^* \beta} \; e^{-\frac{1}{2}|\beta|^2}$$

Aside: Gaussian integrals $\displaystyle \int \frac{d^2\beta}{\pi} e^{-\gamma|\beta|^2} e^{\alpha \beta^* - \alpha^* \beta} = \frac{1}{\gamma} e^{-\frac{|\alpha|^2}{\gamma}}$

$$\Rightarrow \boxed{W(\alpha) = \frac{2}{\pi} e^{-2|\alpha - \alpha_0|^2}}$$

Gaussian centered at $\alpha_0 = \underline{X_0 + i P_0}$

In X, P: $\quad W(X, P) = \frac{1}{2} W\left(\alpha = \frac{X + iP}{\sqrt{2}}\right) = \frac{1}{\pi} e^{-(X - X_0)^2 - (P - P_0)^2} \qquad \Delta X = \Delta P = \frac{1}{\sqrt{2}}$



When we drawn the uncertainty bubble, we can think of this as a countour of the covariance of a Gaussian.

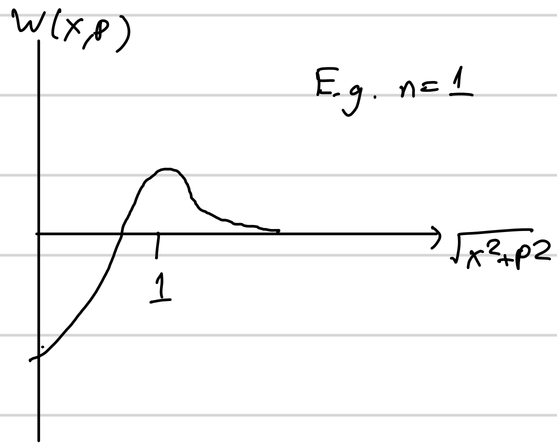
- Fock State $|\psi\rangle = |n\rangle$

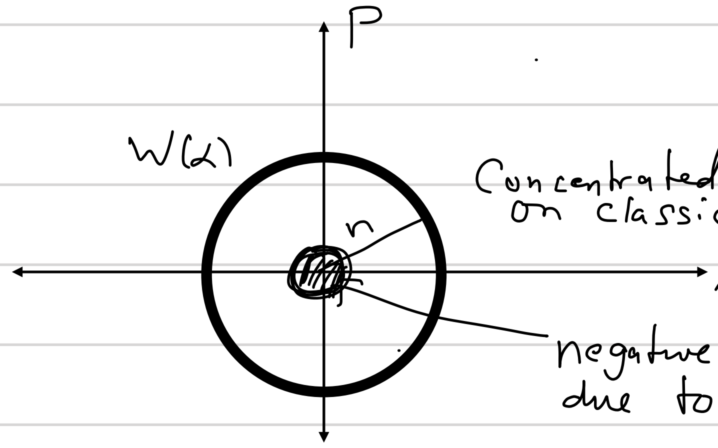
$$\chi(\beta) = \frac{1}{\pi}\langle n|\hat{D}(\beta)|n\rangle = \frac{1}{\pi} e^{-|\beta|^2/2}\,\mathcal{L}_n(|\beta|^2)$$

← Laguerre polynomial (see homework)

$$\Rightarrow W(\alpha) = \int \frac{d^2\beta}{\pi}\, e^{\alpha\beta^* - \alpha^*\beta}\,\chi(\beta) = \frac{2(-1)^n}{\pi}\, e^{-2|\alpha|^2}\,\mathcal{L}_n(4|\alpha|^2)$$

$$W(x,p) = \frac{1}{\pi}(-1)^n\, e^{-(x^2+p^2)}\,\mathcal{L}_n(2(X^2+Y^2))$$





Note: $-\frac{2}{\pi} \leq W(\alpha) \leq \frac{2}{\pi}$. This is generally true, but won't prove it.

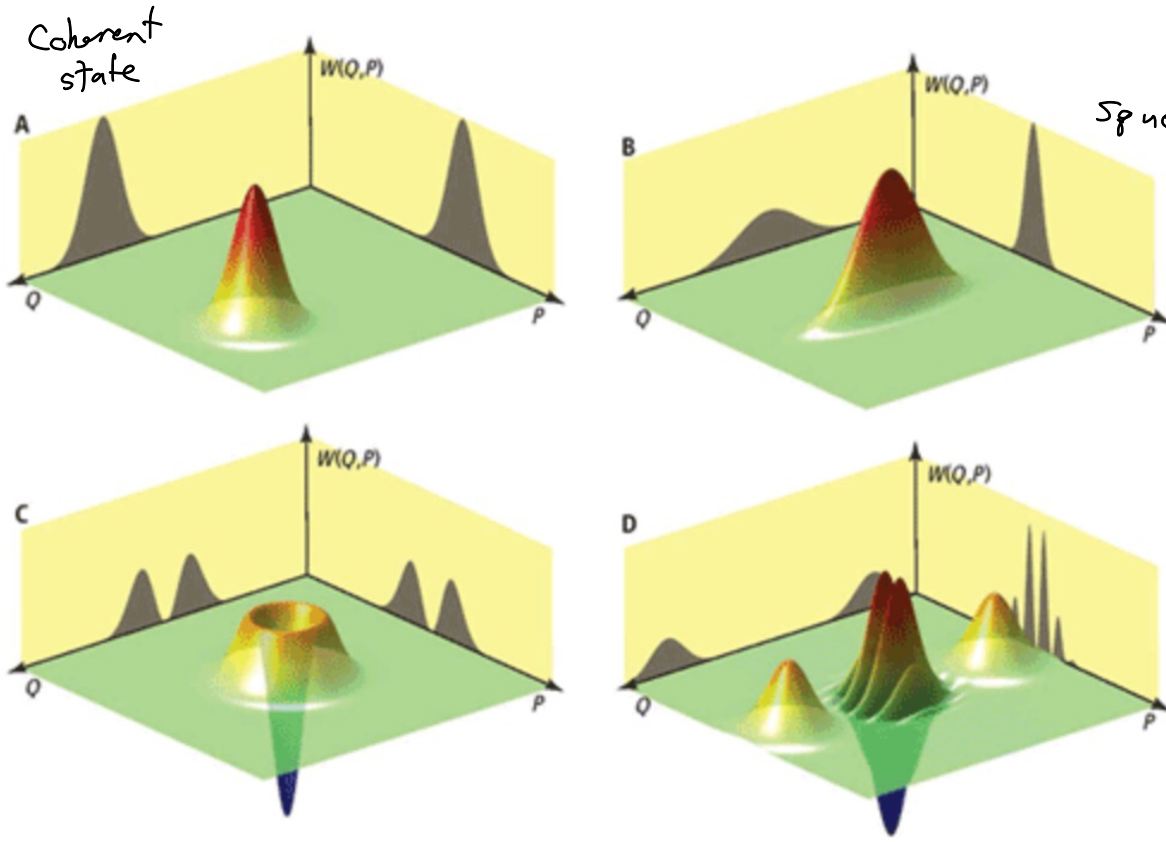
Below are surface plots of the Wigner function for common examples

Coherent state

Squeezed state

n=1 Fock

"Schrödinger cat" state



Taken from: P. Grangier, Science <u>332</u> 313 (2011)

Other Phase Space Representations

While the Wigner function is natural as a symmetrically ordered representation, there are other phase space representations associated with different operator orderings, which also have utility in understanding the quantum-classical boundary.

s-ordered representations

Consider an operator $\hat{A} = f(\hat{a}, \hat{a}^\dagger)$ we can formally express $\hat{A}$ as a power series in powers of $\hat{a}$ and $\hat{a}^\dagger$, but since they don't commute, the expansion coefficients will be different depending on the operator ordering. Three common examples are.

$$\hat{A} = f(\hat{a}, \hat{a}^\dagger) = \sum_{n,m} c^{+1}_{nm} (\hat{a}^\dagger)^n (\hat{a})^m \quad \text{(normal order)}$$

$$\hat{A} = f(\hat{a}, \hat{a}^\dagger) = \sum_{n,m} c^{0}_{nm} \{(\hat{a}^\dagger)^n \hat{a}^m\}_0 \quad \text{(symmetric order)}$$

$$\hat{A} = f(\hat{a}^\dagger, \hat{a}) = \sum_{n,m} c^{-1}_{nm} \hat{a}^m \hat{a}^{\dagger n} \quad \text{(antinormal order)}$$

All three are the same operator, but the coefficients $c^s_{n,m}$ ($s = +1, 0, -1$) (normal, symmetric, antinormal) are different, Symmetric order implies all permutations: e.g. $\hat{a}^\dagger \hat{a} = \frac{\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger}{2} = \{\hat{a}^\dagger \hat{a}\}_s$

$$\boxed{\text{We now defined the } \underline{\text{s-ordered representation}}: \quad A_s(\alpha, \alpha^*) = \sum_{n,m} c^s_{nm} (\alpha^*)^n \alpha^m}$$

Note, there are different conversions. Some authors refer to this as $A_{-s}(\alpha, \alpha^*)$ for reasons we will see. Each of these representations is different.

Clearly, $A_0(\alpha, \alpha^*) = A_W(\alpha, \alpha^*)$ the Weyl symbol, since the Weyl symbol of $\{(\hat{a}^\dagger)^n \hat{a}^m\}_0$ is $(\alpha^*)^n \alpha^m$.

Thus $W(\alpha) = \frac{1}{\pi} \rho_0(\alpha, \alpha^*)$ (Wigner function).

We define $Q(\alpha) \equiv \frac{1}{\pi} \rho_{+1}(\alpha, \alpha^*)$ (Husmi function)

$$P(\alpha) \equiv \frac{1}{\pi} \rho_{-1}(\alpha, \alpha^*) \quad \text{(Glauber – Sudarshan P-function)}$$

We also never actually due the operator ordering by hand. We can find these functions more directly.

Generating function of ordered-products

Consider $\hat{D}(\beta) = e^{\beta\hat{a}^\dagger - \beta^*\hat{a}} = \sum_n \frac{(\beta\hat{a}^\dagger - \beta^*\hat{a})^n}{n!} = \sum_{n,m} \frac{\beta^n (-\beta^*)^m}{n!\, m!} \{\hat{a}^{\dagger n} \hat{a}^m\}_0$

The symmetrically ordered displacement operator is the generator of a symmetrically ordered power series.

Now we have $\hat{A} = \int \frac{d^2\beta}{\pi} \mathrm{Tr}(\hat{D}^\dagger(\beta)\hat{A})\, \hat{D}(\beta) = \sum_{n,m} c^0_{nm} \{\hat{a}^{\dagger n} \hat{a}^m\}_0$

$$\Rightarrow c^0_{nm} = \int \frac{d^2\beta}{\pi} \frac{\beta^n (-\beta^*)^m}{n!\, m!} \mathrm{Tr}(\hat{D}(\beta)\hat{A})$$

$$A_0(\alpha,\alpha^*) = \sum_{n,m}{}' c^0_{nm} (\alpha^*)^n \alpha^m = \int \frac{d^2\beta}{\pi} \sum_n \frac{(\alpha^*\beta)^n}{n!} \sum_m \frac{(\alpha\beta^*)^m}{n!} \mathrm{Tr}(\hat{D}(\beta)\hat{A})$$

$$= \int \frac{d^2\beta}{\pi} e^{\alpha^*\beta - \alpha\beta^*} \mathrm{Tr}(\hat{D}(\beta)\hat{A}) = \mathrm{Tr}(\hat{T}(\alpha)\hat{A}) = \mathrm{Tr}(\delta(\hat{a}-\alpha, \hat{a}^\dagger-\alpha^*)\hat{A})$$

Weyl symbol!

We obtain other operator orderings as follows

Normal $:\hat{D}(\beta): \equiv e^{\beta\hat{a}^\dagger} e^{-\beta^*\hat{a}} = \sum_{n,m} \frac{\beta^n (-\beta^*)^m}{n!\, m!} (\hat{a}^\dagger)^n (\hat{a})^m$

Note: $\hat{D}(\beta) = e^{-\frac{|\beta|^2}{2}} e^{\beta\hat{a}^\dagger} e^{-\beta^*\hat{a}} = e^{-|\beta|^2/2} :\hat{D}(\beta): \Rightarrow :\hat{D}(\beta): = e^{+\frac{|\beta|^2}{2}} \hat{D}(\beta)$

Antinormal: $\cdot\cdot\hat{D}(\beta)\cdot\cdot = e^{-\beta^*\hat{a}} e^{\beta\hat{a}^\dagger} = \sum_{nm} \frac{\beta^n (-\beta^*)^m}{n!\, m!} (\hat{a})^m (\hat{a}^\dagger)^n$

$\hat{D}(\beta) = e^{+\frac{|\beta|^2}{2}} e^{-\beta^*\hat{a}} e^{\beta\hat{a}^\dagger} = e^{+\frac{|\beta|^2}{2}} \cdot\cdot\hat{D}(\beta)\cdot\cdot \Rightarrow \cdot\cdot\hat{D}(\beta)\cdot\cdot = e^{-\frac{|\beta|^2}{2}} \hat{D}(\beta)$

Generally we define s-ordered $\hat{D}_s(\beta) = e^{s|\beta|^2/2} \hat{D}(\beta)$ $s = -1, 0, +1$

Now $\int \frac{d^2\beta}{\pi} |\hat{D}(\beta))(\hat{D}(\beta)| = \int \frac{d^2\beta}{\pi} |\hat{D}_s(\beta))(\hat{D}_{-s}(\beta)|$ since $e^{\frac{s|\beta|^2}{2}} e^{\frac{-s|\beta|^2}{2}} = 1$

Since $\hat{D}_s(\beta)$ is s-order, we obtain the coeffients c^s_{nm} as before

$$|\hat{A}) = \int \frac{d^2\beta}{\pi}\, |\hat{D}_s(\beta))\,(\hat{D}_{-s}(\beta)|\hat{A}) \Rightarrow \hat{A} = \int \frac{d^2\beta}{\pi} \mathrm{Tr}(\hat{D}^\dagger_{-s}(\beta)\,\hat{A})\,\hat{D}_s(\beta) = \sum_{n,m} C^s_{nm} \{\hat{a}^{\dagger n} \hat{a}^m\}_s$$

$$\Rightarrow C^s_{n,m} = \int \frac{d^2\beta}{\pi} \frac{\beta^n(-\beta^*)^m}{n!\,m!} \mathrm{Tr}(\hat{A}\,\hat{D}_{-s}(\beta))$$

$$\Rightarrow A_s(\alpha,\alpha^*) = \int \frac{d^2\beta}{\pi} e^{\alpha\beta^* - \alpha^*\beta}\, \mathrm{Tr}(\hat{A}\,\hat{D}_{-s}(\beta))$$

$$\boxed{A_s(\alpha,\alpha^*) = \mathrm{Tr}(\hat{A}\,\hat{T}_{-s}(\alpha))}$$

Note the opposite sign!

Where I defined the s-order delta function

$$\hat{T}_s(\alpha) = \int \frac{d^2\beta}{\pi} e^{\alpha\beta^* - \alpha^*\beta}\, \hat{D}_s(\alpha) = \pi\,\delta_s(\hat{a}-\alpha, \hat{a}^\dagger - \alpha^*)$$

$$\boxed{\begin{array}{ll} \hat{T}_{+1}(\alpha) = \pi\,\delta(\hat{a}^\dagger - \alpha^*)\,\delta(\hat{a}-\alpha) & : \text{Normally ordered delta function} \\ \hat{T}_{-1}(\alpha) = \pi\,\delta(\hat{a}-\alpha)\,\delta(\hat{a}^\dagger - \alpha^*) & : \text{Antinormally ordered delta function} \end{array}}$$

From completeness $\int \frac{d^2\beta}{\pi} |\hat{D}_s(\beta))\,(\hat{D}_{-s}(\beta)| = \int \frac{d^2\alpha}{\pi} |\hat{T}_s(\alpha))\,(\hat{T}_{-s}(\alpha)|$

$$\Rightarrow \langle\hat{A}\rangle = \mathrm{Tr}(\hat{\rho}\,\hat{A}) = (\hat{\rho}|\hat{A}) = \int \frac{d^2\alpha}{\pi} (\hat{\rho}|\hat{T}_s(\alpha))\,(\hat{T}_{-s}(\alpha)|\hat{A})$$

$$= \int \frac{d^2\alpha}{\pi} \rho_{-s}(\alpha,\alpha^*)\, A_s(\alpha,\alpha)$$ Dual pairing!

$$\boxed{\langle\hat{A}\rangle = \int d^2\alpha\, W_{-s}(\alpha)\, A_s(\alpha)}$$

Note the dual pairing!

$W_s(\alpha) = \frac{1}{\pi}\rho_s(\alpha,\alpha)$ $\quad W_0(\alpha) = $ Wigner, $W_{-1}(\alpha) = P(\alpha)$, $W_{+1}(\alpha) = Q(\alpha)$

$$\hat{A} = \int \frac{d^2\alpha}{\pi} A_s(\alpha,\alpha^*)\,\hat{T}_s(\alpha)$$

Note: $\hat{T}_{-1}(\alpha) = \pi\,\delta(\hat{a}-\alpha)\,\delta(\hat{a}^\dagger - \alpha^*) = \pi \int \frac{d^2\gamma}{\pi} \delta(\hat{a}-\alpha)|\gamma\rangle\langle\gamma|\,\delta(\hat{a}^\dagger - \alpha^*)$

$$= \int d^2\gamma\, \delta^{(2)}(\gamma-\alpha)\, |\gamma\rangle\langle\gamma| = \boxed{|\alpha\rangle\langle\alpha| = \hat{T}_{-1}(\alpha)}$$

$$\therefore Q(\alpha) = \frac{1}{\pi}\mathrm{Tr}(\hat{\rho}\,\hat{T}_{-1}(\alpha)) = \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle = \frac{1}{\pi}|\langle\alpha|\psi\rangle|^2 \geq 0$$ everywhere

The Husimi representation is everywhere positive and thus represents a true probability distribution. It is the probability of measurement outcomes associated with the POVM

$$\int d^2\alpha \, \frac{|\alpha\rangle\langle\alpha|}{\pi} = \hat{\mathbb{1}}$$

While true, it is not generally useful for predicting the measurement outcome of any other POVM (as we will see)

Note also $\hat{\rho} = \int d^2\alpha \; W_s(\alpha)\, \hat{T}_s(\alpha) = \int d^2\alpha \; W_{-1}(\alpha)\, \hat{T}_{-1}(\alpha)$

$$\boxed{\hat{\rho} = \int d^2\alpha \; P(\alpha)\, |\alpha\rangle\langle\alpha|}$$

Thus, if the P-function is positive, $\hat{\rho}$ is a statistical mixture of coherent states.

Existence: So far we have not actually proven that any of the operator ordered power series actually converge. To prove this we turn to the Fourier analysis.

The class of function for which the Fourier transform exists are the Square normalizable functions

$$\|f\|^2 \equiv \int \frac{d^2\alpha}{\pi} \, |f(\alpha)|^2 \; : \text{Finite}$$

If $\|f\|$ is finite $\tilde{f}(\beta) = \int \frac{d^2\alpha}{\pi} \, f(\alpha)\, e^{\alpha\beta^* - \alpha^*\beta}$ exists, and

$$\|f\|^2 = \int \frac{d^2\beta}{\pi} \, |\tilde{f}(\beta)|^2$$

Bound operators: $\|\hat{A}\|^2 \equiv (\hat{A}|\hat{A}) = \mathrm{Tr}(\hat{A}^\dagger\hat{A}) = \int \frac{d^2\alpha}{\pi} \, \hat{A}_s^*(\alpha)\, \hat{A}_{-s}(\alpha)$

$$\|\hat{A}\|^2 = \int \frac{d^2\alpha}{\pi} \, |A_0(\alpha)|^2 = \int \frac{d^2\beta}{\pi} \, |\tilde{A}(\beta)|^2 \; : \text{Bounded if finite}$$

Wigner function (W representation)

$\hat{\rho}$ is a bounded operator $\Rightarrow \mathrm{Tr}(\hat{\rho}^2) = \int \frac{d^2\beta}{\pi} |\chi_0(\beta)|^2 \leq 1$

$\Rightarrow$ The characteristic function of Wigner function is in $\mathcal{L}_2(\mathbb{R}^2)$

$\Rightarrow$ $\boxed{W(\alpha) \text{ always exists for a physical state}}$

More generally the symmetrically ordered Weyl symbol $\hat{A}_0(\alpha)$ always exists

Husimi distribution (Q-representation)

$$Q(\alpha) = W_{+1}(\alpha) = \int \frac{d^2\beta}{\pi} \chi_{+1}(\beta)\, e^{\alpha\beta^* - \alpha^*\beta}$$

$$\chi_{+1}(\beta) = \frac{1}{\pi}\mathrm{Tr}(\hat{\rho}\,\hat{D}_{-1}(\beta)) = \frac{e^{-|\beta|^2/2}}{\pi} \chi_0(\beta)$$

$\Rightarrow$ The characteristic function of Q is always square normalizable

$\Rightarrow$ $\boxed{Q(\alpha) \text{ always exists for a physical state}}$

More generally the normally ordered Weyl symbol $\hat{A}_{+1}(\alpha)$ always exists

Glauber distribution (P-representation)

$$P(\alpha) = W_{-1}(\alpha) = \int \frac{d^2\beta}{\pi} \chi_{-1}(\beta)\, e^{\alpha\beta^* - \alpha^*\beta}$$

$$\chi_{-1}(\beta) = \frac{1}{\pi}\mathrm{Tr}(\hat{\rho}\,\hat{D}_{+1}(\beta)) = \frac{e^{+|\beta|^2/2}}{\pi} \chi_0(\beta) : \text{Generally \underline{unbounded}}$$

$\Rightarrow$ $\boxed{P(\alpha) \text{ Only exists if } \chi_0(\beta) \text{ falls of at least as fast as fast as } e^{-|\beta|^2/2}.}$

More generally, the antisymmetrically ordered Weyl symbol $\hat{A}_{-1}(\alpha)$ doesnt exist as a "tempered" function.

Relations between P, Q, W

$$W(\alpha) = \frac{2}{\pi}\int P(\beta)\, e^{-2|\beta-\alpha|^2}\, d^2\beta$$

$$Q(\alpha) = \frac{2}{\pi}\int W(\beta)\, e^{-2|\beta-\alpha|^2}\, d\beta = \frac{1}{\pi}\int P(\beta)\, e^{-|\alpha-\beta|^2}\, d\beta$$

$\Rightarrow$ W is a "smoothed" version of P, Q is a "smoothed" version of W. Q is the most "coarse grained" (least rippled).

Another useful formula for calculating $P(\alpha)$ and $W(\alpha)$

Given $\hat{\rho} = \int d^2\alpha\, P(\alpha)\, |\alpha\rangle\langle\alpha| \Rightarrow \langle -\beta|\hat{\rho}|\beta\rangle = \int d^2\alpha\, P(\alpha)\, \langle -\beta|\alpha\rangle\langle\alpha|\beta\rangle$

$$\Rightarrow \langle -\beta|\hat{\rho}|\beta\rangle = \int d^2\alpha\, P(\alpha)\, e^{-|\alpha|^2-|\beta|^2}\, e^{\alpha\beta^*-\alpha^*\beta}$$

$$\Rightarrow \langle -\beta|\hat{\rho}|\beta\rangle \frac{e^{|\beta|^2}}{\pi} = \int \frac{d^2\alpha}{\pi}\left(P(\alpha)\, e^{-|\alpha|^2}\right) e^{\alpha\beta^*-\alpha^*\beta} \qquad \text{(2D Fourier transform)}$$

$$\Rightarrow \boxed{P(\alpha) = \frac{e^{|\alpha|^2}}{\pi^2}\int d^2\beta\, \langle -\beta|\hat{\rho}|\beta\rangle\, e^{|\beta|^2}\, e^{\alpha\beta^*-\alpha^*\beta}}$$

$$W(\alpha) = \frac{2}{\pi}\int P(\beta')\, e^{-2|\beta'-\alpha|^2}\, d^2\beta'$$

$$\Rightarrow \boxed{W(\alpha) = 2\,\frac{e^{2|\alpha|^2}}{\pi^2}\int d^2\beta\, \langle -\beta|\hat{\rho}|\beta\rangle\, e^{2(\alpha\beta^*-\alpha^*\beta)}}$$

Dynamics

The classical (Hamiltonian) dynamics is described by the Liouville eqn.

$$\frac{\partial}{\partial t} P(x,p,t) = \{H(x,p), P(x,p,t)\}_{P.B.} = \frac{\partial H}{\partial x}\frac{\partial P}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial P}{\partial x}$$

where "P.B" stands for the Poisson bracket. We see the analogous phase space representation of Schrodinger equation. For example we seek

$$\frac{\partial W(x,p,t)}{\partial t},$$ the time evolution of the Wigner function.

For the purpose we introduce the "star product" (also know as Moyal product)
Let $\hat{C} = \hat{A}\hat{B}$. Then the Weyl symbol $C_w(x,p) \equiv A_w(x,p) \star B_w(x,p)$

Thus, given the Schrödinger equation

$$\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] = \frac{-i}{\hbar}(\hat{H}\hat{\rho} - \hat{\rho}\hat{H})$$ (keeping the ħ's around for now)

$$\Rightarrow \boxed{\frac{\partial W(x,p,t)}{\partial t} = \frac{-i}{\hbar}\left(H_w(x,p) \star W(x,p,t) - W(x,p,t) \star H(x,p)\right) \equiv \{H_w(x,p), W(x,p,t)\}_{M.B.}}$$

Where here, "M.B" stands for Moyal Bracket. The Schrödinger equation for Wigner function has the same form as the classical Liouville equation with the Poisson bracket replaced by the Moyal bracket.

So far this is all a formal statement. We need to find the star product.
Using the symmetric delta: $2\pi\hbar\,\delta(\hat{x}-x, \hat{p}-p) \equiv \hat{T}(x,p) = \int \frac{dx'dp'}{2\pi\hbar} e^{\frac{-i}{\hbar}(\hat{p}-p)x' + \frac{i}{\hbar}(\hat{x}-x)p'}$

and $\hat{A} = \int \frac{dx\,dp}{2\pi\hbar} A_w(x,p)\,\hat{T}(x,p)$, $\hat{B} = \int \frac{dx\,dp}{2\pi\hbar} B_w(x,p)\,\hat{T}(x,p)$, $\hat{C} = \hat{A}\hat{B}$

$$\Rightarrow C_w(x,p) = \mathrm{Tr}(\hat{C}\,\hat{T}(x,p)) = \int \frac{dx'dp'\,dx''dp''}{(2\pi\hbar)^2} A_w(x',p')\, B_w(x'',p'')\, \underbrace{\mathrm{Tr}(\hat{T}(x',p')\,\hat{T}(x'',p'')\,\hat{T}(x,p))}$$

With some algebra: $e^{-\frac{2i}{\hbar}[(p'-p)(x''-x) - (p''-p)(x'-x)]}$

Making the change of variables $x_1 \equiv x' - x$, $p_1 \equiv p' - x$, $x_2 \equiv x'' - x$, $p_2 \equiv p'' - x$

$$\Rightarrow A_w(x,p) \star B_w(x,p) = \int \frac{dx_1\, dp_1\, dx_2\, dp_2}{(2\pi\hbar)^2} A_w(x_1 + x, p_1 + x)\, B_w(x_2 + x, p_2 + p)\, e^{-\frac{2i}{\hbar}(p_1 x_2 - x_1 p_2)}$$

$$= \int \frac{dx_1\, dp_1\, dx_2\, dp_2}{(2\pi\hbar)^2} \left(\sum_{n,m} \frac{1}{n!}\frac{1}{m!} x_1^n p_1^m \partial_x^n \partial_p^m A_w(x,p) \right) \left(\sum_{k,l} \frac{1}{k!}\frac{1}{l!} x_2^k p_2^l \partial_x^k \partial_p^l B_w(x,p) \right) e^{-\frac{2i}{\hbar}(p_1 x_2 - x_1 p_2)}$$

Consider a term in the sum: $\int \frac{dx_1\, dp_1\, dx_2\, dp_2}{(2\pi\hbar)^2} x_1^n p_1^m x_2^k p_2^l e^{-\frac{2i}{\hbar}(p_1 x_2 - x_1 p_2)}$

$$= \left[\int \frac{dx_1\, dp_2}{(2\pi\hbar)} x_1^n \left(\frac{\hbar}{+2i}\partial_{x_1}\right)^l e^{+\frac{2i}{\hbar}x_1 p_2} \right] \left[\int \frac{dx_2\, dp_1}{2\pi\hbar} x_2^k \left(\frac{\hbar}{-2i}\partial_{x_2}\right)^m e^{-\frac{2i}{\hbar}x_2 p_1} \right]$$

$$= \int dx_1\, x_1^n \left(\frac{\hbar}{+2i}\partial_{x_1}\right)^l \delta(x_1) \int dx_2\, x_2^k \left(\frac{\hbar}{-2i}\partial_{x_2}\right)^m \delta(x_2)$$

$$= \left(\frac{\hbar}{-2i}\partial_{x_1}\right)^l x_1^n \Big|_{x_1=0} \left(\frac{\hbar}{+2i}\partial_{x_2}\right)^m x_2^k \Big|_{x_2=0} = \left(\frac{\hbar}{-2i}\right)^n n!\, \left(\frac{\hbar}{+2i}\right)^m m!\, \delta_{ln}\, \delta_{km}$$

$$\Rightarrow A_w(x,p) \star B_w(x,p) = \sum_{n,m} \frac{1}{n!}\frac{1}{m!} \left(\frac{\hbar}{-2i}\partial_x^n \partial_p^m A_w(x,p)\right)\left(\frac{\hbar}{2i}\partial_p^n \partial_x^m B_w(x,p)\right)$$

$$= A_w(x,p) \left[\sum_n \frac{1}{n!}\left(\frac{-\hbar}{2i}\overleftarrow{\partial}_x \overrightarrow{\partial}_p\right)^n \right] \left[\sum_m \frac{1}{m!}\left(\frac{\hbar}{+2i}\overleftarrow{\partial}_p \overrightarrow{\partial}_x\right)^m \right] B_w(x,p)$$

$$\boxed{\begin{aligned} &\Rightarrow A_w(x,p) \star B_w(x,p) = A_w(x,p)\, e^{\frac{i\hbar}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)}\, B_w(x,p) \equiv A_w(x,p)\, e^{\frac{i\hbar}{2}\Lambda}\, B_w(x,p) \\ &A_w \Lambda B_w = A_w(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x) B_w = \partial_x A_w \partial_p B_w - \partial_p A_w \partial_x B_w \end{aligned}}$$

Note $A_w \Lambda^n B_w = -B_w \Lambda^n A_w$

$$\boxed{\begin{aligned} &\Rightarrow \{A_w(x,p), B_w(x,p)\}_{MB} = \frac{1}{i\hbar}\left(A_w e^{\frac{i\hbar}{2}\Lambda} B_w - B_w e^{\frac{i\hbar}{2}\Lambda} A_w\right) = \frac{1}{i\hbar} A_w \left(e^{\frac{i\hbar}{2}\Lambda} - e^{-\frac{i\hbar}{2}\Lambda}\right) B_w \\ &\Rightarrow \underline{\text{Moyal Bracket}}\quad \{A_w(x,p), B_w(x,p)\}_{MB} = \frac{2}{\hbar} A_w(x,p) \sin\left(\frac{\hbar}{2}\Lambda\right) B_w(x,p) \end{aligned}}$$

Note: In the limit "$\hbar \to 0$" (whatever that means)

$$\{A_w, B_w\}_{M.B} \to A_w \Lambda B_w = \frac{\partial A_w}{\partial x}\frac{\partial B_w}{\partial p} - \frac{\partial A_w}{\partial p}\frac{\partial B_w}{\partial x} = \{A_w, B_w\}_{P.B.}$$

The limit $\hbar \to 0$, the Moyal Bracket reduces to the Poisson Bracket!

In particular, the Wigner function evolves according to:

$$\boxed{\begin{aligned}\frac{\partial}{\partial t} W(x,p,t) &= \{H_w(x,p), W(x,p,t)\}_{MB} = \frac{2}{\hbar} H_w(x,p) \sin\left(\frac{\hbar}{2}\Lambda\right) W(x,p,t) \\ &= \underbrace{\{H_w(x,p), W(x,p,t)\}_{P.B.}}_{\text{Classical flow}} + \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n \hbar^{2n}}{2^{2n}(2n+1)!} H_w(x,p)\, \Lambda^{2n+1}\, W(x,p)}_{\text{Quantum corrections to dynamics}}\end{aligned}}$$

If we can neglect the terms proportional to $\hbar^{2n}$, the Wigner function evolves according to the classical flow. Truncating the series at $\hbar = 0$ is known as the truncated Wigner approximation (TWA)

When is the TWA valid? Let's look at simple wave mechanics, with a Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$. The Heisenberg equation impled

$$\frac{d\langle\hat{x}\rangle}{dt} = \frac{\langle\hat{p}\rangle}{m}, \qquad \frac{d\langle\hat{p}\rangle}{dt} = -\left\langle \frac{\partial V(x)}{\partial x}\Big|_{x=\hat{x}} \right\rangle$$

However $\left\langle \frac{\partial V(x)}{\partial x}\Big|_{x=\hat{x}}\right\rangle \neq \frac{\partial V}{\partial x}\Big|_{x=\langle\hat{x}\rangle}$ in general. When this is true (or approximately true) $\langle\hat{x}\rangle$ and $\langle\hat{p}\rangle$ follow the classical trajectories. An example is for a harmonic potential $V(\hat{x}) = \frac{1}{2}m\omega\hat{x}^2$, then

$$\frac{\partial V}{\partial x}\Big|_{x=\hat{x}} = m\omega\hat{x} \qquad \frac{d\langle\hat{x}\rangle}{dt} = \frac{\langle\hat{p}\rangle}{m}, \quad \frac{d\langle\hat{p}\rangle}{dt} = -m\omega\langle\hat{x}\rangle$$

The expectation values in SHO follow the classical flow. For the SHO, the TWA is exact!

What about approximate classical flow? If we have a wavepacket whose extent Δx is small compared to the scale over which $V(x)$ changes, then $\langle \frac{\partial V}{\partial x}\big|_{x=\hat{x}} \rangle \approx \frac{\partial V}{\partial x}\big|_{x=\langle \hat{x} \rangle}$. Moreover we don't want Δx to be too small otherwise Δp will be too large and nonclassical flow will follow. In a nonlinear potential Δx will grow and eventually the classical approximation will break down. The time at which this occurs is known as the <u>Ehrenfest time</u>.

More quantitatively, consider the time evolution of the Wigner function for wave mechanics

$$\frac{\partial W}{\partial t} = \{H_w, W\}_{P.B.} + \sum_{n=1}^{\infty} \frac{(-1)^n \hbar^{2n}}{2^{2n}(2n+1)!} \left(\partial_x^{2n+1} V(x)\right)\left(\partial_p^{2n+1} W(x,p,t)\right)$$

Firstly corrections arise for potentials beyond quadratic, cubic or greater.
Secondly when $W(x,p,t)$ is relatively smooth function of p on some appropriate scale, the higher order dervatives are negligible. We have see that the very rapid oscillations in $W(x,p)$ can arise in nonclassical states, such as the cat state, and these will lead to nonclassical flow. The Ehrenfest time will depend on the scale of features in $W(x,p)$ (characteristic Action I) compared to $\hbar$. As we will see this subplanck scale structure is associated with nonclassicality, and a break down of the TWA.

<u>Other representations</u>

<u>Bopp-representation</u>: As follows from $e^{x_0 \partial_x} f(x) = f(x+x_0)$

$$A_w(x,p) \star B_w(x,p) = A\left(\hat{x} = x + i\tfrac{\hbar}{2}\partial_p,\ \hat{p} = p - i\tfrac{\hbar}{2}\partial_x\right) B_w(x,p)$$

$$= B\left(\hat{x} = x - i\tfrac{\hbar}{2}\partial_p,\ \hat{p} = p + i\tfrac{\hbar}{2}\partial_x\right) A_w(x,p)$$

Thus
$$(\hat{x}\hat{A})_w = x A_w(x,p) + i\tfrac{\hbar}{2}\partial_p A_w(x,p)$$

$$(\hat{A}\hat{x})_w = x A_w(x,p) - i\tfrac{\hbar}{2}\partial_p A_w(x,p)$$

$$[\hat{x}, \hat{A}]_w = i\hbar\, \partial_p A_w = i\hbar \{x, A_w(x,p)\}_{P.B.} \quad \text{as expected}$$

Complex amplitude: (setting $\hbar = 1$) $\alpha = \frac{X+iP}{\sqrt{2}}$, $\frac{\partial}{\partial x} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \alpha^*}\right)$, $\frac{\partial}{\partial P} = \frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \alpha^*}\right)$

$\Rightarrow$ Poisson bracket: $\{A, B\}_{PB} = -i\left(\frac{\partial A}{\partial \alpha}\frac{\partial B}{\partial \alpha^*} - \frac{\partial A}{\partial \alpha^*}\frac{\partial B}{\partial \alpha}\right) = -i\, A\, \Lambda_c\, B$

$\Rightarrow$ Star product: $(\hat{A}\hat{B})_W = A_W(\alpha,\alpha^*)\, e^{\frac{\Lambda_c}{2}}\, B_W(\alpha,\alpha^*)$ $\quad \left(\Lambda_c = \overleftarrow{\partial}_\alpha \overrightarrow{\partial}_{\alpha^*} - \overleftarrow{\partial}_{\alpha^*}\overrightarrow{\partial}_\alpha\right)$

$\Rightarrow$ Moyal bracket: $\{A, B\}_{M.B} = -2 A_W(\alpha,\alpha^*) \sinh\left[\frac{\Lambda_c}{2}\right] B_W(\alpha,\alpha^*)$

$$= \{A_W(\alpha,\alpha^*), B_W(\alpha,\alpha^*)\}_{P.B}$$

$\Rightarrow$ Bopp-representation

$$\left(A(\hat{a},\hat{a}^+)\, B(\hat{a},\hat{a}^+)\right)_W = A\left(\hat{a} = \alpha + \tfrac{1}{2}\partial_{\alpha^*},\ \hat{a} = \alpha^* - \tfrac{1}{2}\partial_\alpha\right) B_W(\alpha,\alpha^*)$$
$$= B\left(\hat{a} = \alpha - \tfrac{1}{2}\partial_{\alpha^*},\ \hat{a}^\dagger = \alpha^* + \tfrac{1}{2}\partial_\alpha\right) A_W(\alpha,\alpha^*)$$

The Bopp-representation is the easiest way to determine the Weyl symbol of normally ordered operators.

E.g. $(\hat{a}^+\hat{a})_W = (\alpha^* + \tfrac{1}{2}\partial_\alpha)\,\alpha = |\alpha|^2 + \tfrac{1}{2}$ as expected

Finally we have the TWA expansion

$$\frac{\partial W}{\partial t} = \{H, W\}_{M.B.} = \underbrace{-i\left(\frac{\partial H_w}{\partial \alpha}\frac{\partial W}{\partial \alpha^*} - \frac{\partial H_w}{\partial \alpha^*}\frac{\partial W}{\partial \alpha}\right)}_{\{H,W\}_{P.B.}\ \ \text{TWA}} \underbrace{- \frac{i}{8}\left(\frac{\partial^3 H_w}{\partial^2\alpha^*\partial\alpha}\frac{\partial^3 W}{\partial^2\alpha\,\partial\alpha^*} - \frac{\partial^3 H_w}{\partial^2\alpha\,\partial\alpha^*}\frac{\partial^3 W}{\partial^2\alpha^*\partial\alpha}\right) + \dots}_{\text{Beyond TWA}}$$

In the complex representation $\hbar = 1$, so subplanck structure in for action (area) $I < 1$.

Example of the SHO

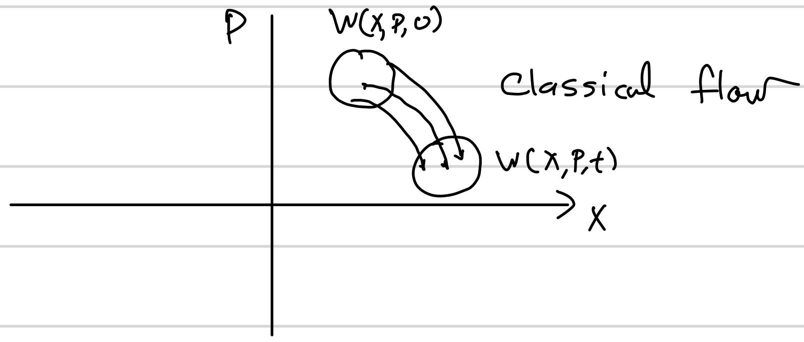
For the SHO $\hat{H} = \omega\left(\frac{\hat{X}^2 + \hat{P}^2}{2}\right) = \omega\left(\frac{\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger}{2}\right)$ $(\hbar = 1)$ $\Rightarrow$ $H_W(X,P) = \omega\left(\frac{X^2+P^2}{2}\right) = \omega\alpha^*\alpha$

$\Rightarrow$ TWA is exact: $\frac{\partial W}{\partial t} = i\omega\left(\alpha^*\frac{\partial W}{\partial \alpha^*} - \alpha\frac{\partial W}{\partial \alpha}\right) = \omega\left(X\frac{\partial}{\partial P} - P\frac{\partial}{\partial X}\right) W$

Note: $X\frac{\partial}{\partial P} - P\frac{\partial}{\partial X} = -\frac{\partial}{\partial \phi}$ where $X + iP = re^{i\phi}$

= generator of rotation in phase space (as expected)

$\frac{\partial}{\partial t}W(r,\phi,t) = -\omega\frac{\partial}{\partial\phi}W(r,\phi,t) \Rightarrow W(r,\phi,t) = W(r,\phi-\omega t,0)$



As the TWA is exact, the solution for the Wigner function is the classical solution $W(X,P,t) = W(X(-t),P(-t),0)$. Given an initial Gaussian state, e.g. a coherent state, $W(X,P,0) = \frac{1}{\pi}e^{-(X-X_0)^2-(P-P_0)^2}$, $W(\alpha,0) = \frac{2}{\pi}e^{-2|\alpha-\alpha_0|^2}$

$$\Rightarrow W(\alpha,t) = \frac{2}{\pi}e^{-2|\alpha e^{i\omega t}-\alpha_0|^2} = \frac{2}{\pi}e^{-2|\alpha-\alpha_0 e^{-i\omega t}|^2}$$

Gaussian state $\Rightarrow$ Gaussian state

Generally, the TWA is exact where $\hat{H}$ has no more than quadratic terms in $\hat{a}$ and $\hat{a}^\dagger$. Then $H_W(\alpha,\alpha^*)$ will be no more than quadric in α + α^*. For this case the flow is classical. If we start with a pure state with a positive Wigner function, a "Gaussian state" it will remain Gaussian. Nonclassical dynamics occurs for anharmonic oscillators, with terms beyond quadric in $\hat{a}$ and $\hat{a}^\dagger$. The resulting nonclassical flow results in nonGaussian states with a negative Wigner function and sub Planck scale structure.

Our goal with be to study the open quantum system dynamics and see how and when decoherence recovers the classical limit.