Physics 581: Open Quantum Systems

Lecture 11: Phase space representation of master equation

For the damped simple harmonic oscillator we can consider the phase space representations, P, Q, W. This will give us the most direct comparison between the quantum features of open systems and the classical features. Of particular interest to us is how and when decoherence supresses quantum features and helps us to understand the emergence of the classical world.

We consider the master equation for the damped SHO

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] + \mathcal{L}[\hat{\rho}]$$

where $\mathcal{L}[\hat{\rho}] = -\frac{\Gamma}{2}(\bar{n}+1)(\hat{a}^\dagger\hat{a}\hat{\rho} + \hat{\rho}\hat{a}^\dagger\hat{a}) + \Gamma(\bar{n}+1)\hat{a}\hat{\rho}\hat{a}^\dagger - \frac{\Gamma}{2}\bar{n}(\hat{a}\hat{a}^\dagger\hat{\rho} + \hat{\rho}\hat{a}\hat{a}^\dagger) + \Gamma\bar{n}\hat{a}^\dagger\hat{\rho}\hat{a}$

Let us first consider the Wigner function and at zero temperature

$$\frac{\partial}{\partial t}W(\alpha,\alpha^*,t) = \underset{\substack{\uparrow\\ \text{Moyal Bracket}}}{\{H_w(\alpha,\alpha^*), W(\alpha,\alpha^*,t)\}_{MB}} + \underbrace{\mathcal{L}[\hat{\rho}]_W}_{\text{Weyl symbol}}$$

Now $\mathcal{L}[\hat{\rho}]_W = -\frac{\Gamma}{2}\left((\hat{a}^\dagger\hat{a})_W \star W + W \star (\hat{a}^\dagger\hat{a})_W\right) + \Gamma(\hat{a}\hat{\rho}\hat{a}^\dagger)_W$

($\star$: Star product)

Using the Bopp representation of the Star product

$$\hat{a} \to \alpha + \tfrac{1}{2}\overrightarrow{\partial}_{\alpha^*} = \alpha - \tfrac{1}{2}\overleftarrow{\partial}_{\alpha^*} \qquad \hat{a}^\dagger = \alpha^* - \tfrac{1}{2}\overrightarrow{\partial}_{\alpha} = \alpha^* + \tfrac{1}{2}\overleftarrow{\partial}_{\alpha}$$

$$\Rightarrow (\hat{a}^\dagger\hat{a})_W \star W = (\alpha^* - \tfrac{1}{2}\partial_\alpha)(\alpha + \tfrac{1}{2}\partial_{\alpha^*})W = (\alpha^* - \tfrac{1}{2}\partial_\alpha)(\alpha W + \tfrac{1}{2}\partial_{\alpha^*}W) = |\alpha|^2 W - \tfrac{1}{2}\partial_\alpha(\alpha W) + \tfrac{\alpha^*}{2}\partial_{\alpha^*}W - \tfrac{1}{4}\frac{\partial^2 W}{\partial\alpha\partial\alpha^*}$$

$$= |\alpha|^2 W - \tfrac{1}{2}\partial_\alpha(\alpha W) + \tfrac{1}{2}\partial_{\alpha^*}(\alpha^* W) - \tfrac{1}{4}\frac{\partial^2 W}{\partial\alpha\partial\alpha^*} - \tfrac{1}{2}W$$

$$W \star (\hat{a}^\dagger\hat{a})_W = W(\alpha)(\alpha^* + \tfrac{1}{2}\overleftarrow{\partial}_\alpha)(\alpha - \tfrac{1}{2}\overleftarrow{\partial}_{\alpha^*}) = (\alpha - \tfrac{1}{2}\partial_{\alpha^*})(\alpha^* W + \tfrac{1}{2}\partial_\alpha W)$$

$$= |\alpha|^2 W - \tfrac{1}{2}\partial_{\alpha^*}(\alpha^* W) + \tfrac{\alpha}{2}\partial_\alpha W - \tfrac{1}{4}\frac{\partial^2 W}{\partial\alpha\partial\alpha^*} = |\alpha|^2 W - \tfrac{1}{2}\partial_{\alpha^*}(\alpha^* W) + \tfrac{1}{2}\partial_\alpha(\alpha W) - \tfrac{1}{4}\frac{\partial^2 W}{\partial\alpha\partial\alpha^*} - \tfrac{1}{2}W$$

$$(\hat a \hat\rho \hat a^\dagger)_W = \left(\alpha + \tfrac12 \overrightarrow{\partial}_{\alpha^*}\right) W \left(\alpha^* + \tfrac12 \overleftarrow{\partial}_\alpha\right) = \left(\alpha^* + \tfrac12\partial_\alpha\right)\left(\alpha W + \tfrac12 \partial_{\alpha^*} W\right)$$

$$= |\alpha|^2 W + \tfrac12 \partial_\alpha(\alpha W) + \tfrac{\alpha^*}{2}\partial_{\alpha^*} W + \tfrac14 \frac{\partial^2 W}{\partial\alpha\partial\alpha^*}$$

$$= |\alpha|^2 W + \tfrac12\partial_\alpha(\alpha W) + \tfrac12 \partial_{\alpha^*}(\alpha^* W) + \tfrac14 \frac{\partial^2 W}{\partial\alpha\partial\alpha^*} - \tfrac12 W$$

Putting it all together

$$\Rightarrow \quad \mathcal{L}[\hat\rho]_W = \frac{\Gamma}{2}\left[\partial_\alpha(\alpha W) + \partial_{\alpha^*}(\alpha W)\right] + \frac{\Gamma}{2}\frac{\partial^2 W}{\partial\alpha^*\partial\alpha} \qquad (\bar n = 0)$$

$\Rightarrow$ With a little more work we can show that at finite temperature

$$\boxed{\mathcal{L}[\hat\rho]_W = \frac{\Gamma}{2}\left[\partial_\alpha(\alpha W) + \partial_{\alpha^*}(\alpha^* W)\right] + \Gamma\left(\bar n + \tfrac12\right)\frac{\partial^2 W}{\partial\alpha\partial\alpha^*}}$$

To understand this equation more clearly, transform to quadrature variables: $\alpha = \frac{1}{\sqrt2}(X + iP)$

$X = \frac{\alpha+\alpha^*}{\sqrt2}$, $P = \frac{\alpha-\alpha^*}{i\sqrt2}$

$$\Rightarrow \quad \frac{\partial}{\partial\alpha} = \frac{\partial X}{\partial\alpha}\frac{\partial}{\partial X} + \frac{\partial P}{\partial\alpha}\frac{\partial}{\partial P} = \frac{1}{\sqrt2}\left(\frac{\partial}{\partial X} - i\frac{\partial}{\partial P}\right), \quad \frac{\partial}{\partial\alpha^*} = \frac{1}{\sqrt2}\left(\frac{\partial}{\partial X} + i\frac{\partial}{\partial P}\right)$$

$$\Rightarrow \quad \boxed{\mathcal{L}[\hat\rho]_W = \frac{\Gamma}{2}\left(\frac{\partial}{\partial X}(XW) + \frac{\partial}{\partial P}(PW)\right) + \frac{\Gamma(\bar n + \frac12)}{2}\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial P^2}\right)W}$$

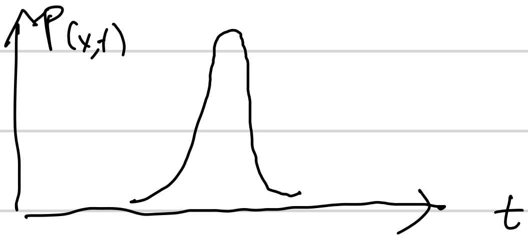
This final equation is a form of the Fokker-Planck equation, well known in classical statistical physics, particularly in the context of Brownian motion.

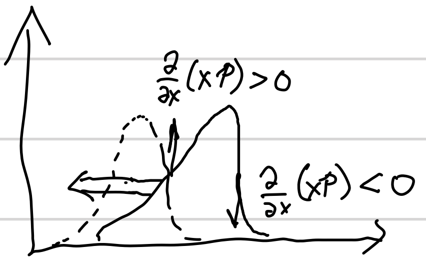
To get a feeling for the Fokker-Planck equation, consider a general form

$$\frac{\partial}{\partial t}P(x,t) = M_1\frac{\partial}{\partial x}\left(x P(x,t)\right) + \frac12 M_2 \frac{\partial^2}{\partial x^2}\left(x^2 P(x,t)\right)$$

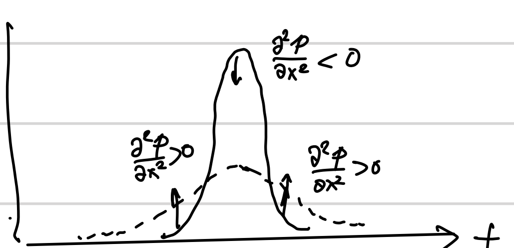
where $P(x,t)$ is a probability distribution of one random variable X at time t.

Suppose at time t





Drift: M_1 = Drift coeffiun



Diffusion $M_2 \equiv$ Diffusion coefficient

The effect of drift and diffusion is made clear by considering the evolution of the first two moments

$$\frac{d}{dt}\langle x\rangle(t) = \int dx\, x\, \frac{\partial P(x,t)}{\partial t} = \int dx\, x\left(M_1 \frac{\partial}{\partial x}(xP) + \frac{M_2}{2}\frac{\partial^2}{\partial x^2}P\right)$$

$$\underset{\substack{\uparrow \\ \text{integration by parts}}}{=} -M_1 \int dx\, x\, P(x,t) = -M_1\langle x\rangle \Rightarrow \boxed{\begin{array}{l}\langle x\rangle(t) = \langle x\rangle(0)\, e^{-M_1 t} \quad \text{(exponential decay)} \\ M_1 = \text{Drift coefficient}\end{array}}$$

$$\frac{d}{dt}\langle\Delta x^2\rangle = \frac{d^2}{dt^2}\left(\langle x^2\rangle - \langle x\rangle^2\right) = \frac{d^2}{dt^2}\langle x^2\rangle - 2\langle x\rangle\frac{d}{dt}\langle x\rangle$$

$$\frac{d}{dt}\langle x^2\rangle = \int dx\, x^2 \frac{\partial P}{\partial t} = \int dx\, x^2\left(M_1\frac{\partial}{\partial x}(xP) + \frac{M_2}{2}\frac{\partial^2}{\partial x^2}P\right)$$

$$= -2M_1\langle x^2\rangle + M_2 \quad \text{(integration by parts)}$$

$$\Rightarrow \frac{d}{dt}\langle\Delta x^2\rangle = -2M_1\langle x^2\rangle + M_2 + 2\langle x\rangle(M_1\langle x\rangle) = -2M_1\langle\Delta x^2\rangle + M_2$$

$$\Rightarrow \boxed{\begin{array}{l}\langle\Delta x^2\rangle(t) = \langle\Delta x^2\rangle(0)\, e^{-2M_1 t} + \left(1 - e^{-2M_1 t}\right)\dfrac{M_2}{2M_1} \\ \text{Steady state } \langle\Delta x^2\rangle \to \dfrac{M_2}{2M_1}\end{array}}$$

Note: For $M_1 t \ll 1$, $\langle\Delta x^2\rangle(t) \simeq \langle\Delta x^2\rangle(0) + M_2 t$

$\Rightarrow$ Time dependent variance grows linearly in time $M_2 t$, Diffussion!

Diffussion Equation: $\frac{\partial}{\partial t}P = \frac{1}{2}D\frac{\partial^2 P}{\partial x^2}$ (Discribes Weiner Stochastic Process : Random Walk)

We can solve for the Green's function.

First let us find the Characteristic function $\chi(k,t) = \int \frac{dx}{\sqrt{2\pi}} P(x,t)\, e^{-ikx}$

$$P(x,t) = \int \frac{dk}{\sqrt{2\pi}}\, \chi(k,t)\, e^{ikx}$$

$$\frac{\partial}{\partial t}\chi(k,t) = -\frac{k^2 D}{2}\chi \Rightarrow \chi(k,t) = \chi(k,0)\, e^{-\frac{k^2 D t}{2}}$$

$$\Rightarrow P(x,t) = \int\frac{dk}{\sqrt{2\pi}}\chi(k,0)\, e^{-\frac{k^2Dt}{2}} e^{ikx} = \int dx'\, P(x',0)\underbrace{\int\frac{dk}{2\pi} e^{-\frac{k^2Dt}{2}} e^{ik(x-x')}}_{G(x-x',t)}$$

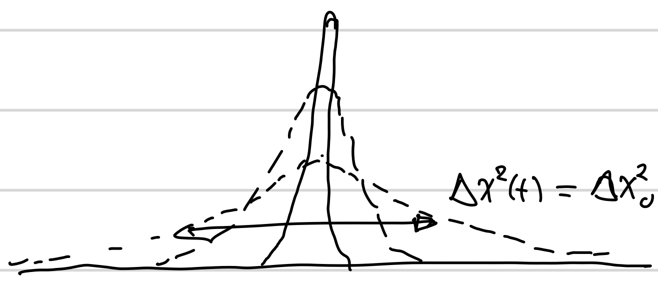
$G(x-x', t)$ is the solution to the diffusion equation when $P(x,0) = \delta(x-x')$

$$G(x-x',t) = \frac{1}{\sqrt{2\pi D t}} e^{-\frac{(x-x')^2}{2Dt}}$$ Gaussian, variance Dt

$\Rightarrow$ General solution to the diffusion equation

$$\boxed{P(x,t) = \frac{1}{\sqrt{2\pi D t}} \int dx' \, e^{-\frac{(x-x')^2}{2Dt}} P(x',0)}$$

Note, if $P(x,0) = \frac{1}{\sqrt{2\pi \Delta x_0^2}} e^{-\frac{(x-x_0)^2}{2\Delta x_0^2}} \Rightarrow P(x,t) = \frac{1}{\sqrt{2\pi \Delta x^2(t)}} e^{-\frac{(x-x_0)^2}{2\Delta x^2(t)}}$ $\quad \Delta x^2(t) = \Delta x_0^2 + Dt$



$\Delta x^2(t) = \Delta x_0^2 + Dt$ (Diffusion)

The Fokker-Planck equation includes both drift and diffusion (Ornstein-Uhlenbeck Process)

The Green's function can be found in the same way as for the diffusion equation. It is also a Gaussian $G(x-x', t) = \frac{1}{\sqrt{2\pi \sigma^2(t)}} e^{-\frac{(x - x' e^{-M_1 t})^2}{2\sigma^2(t)}}$ $\quad \sigma^2(t) = \frac{M_2}{2M_1}(1-e^{-2M_1 t})$

If the initial distribution is Gaussian $P(x,0) = \frac{1}{\sqrt{2\pi \Delta x_0^2}} e^{-\frac{(x-x_0)^2}{2\Delta x_0^2}}$

$$\Rightarrow P(x,t) = \int dx' \, G(x-x', t) P(x,0) = \frac{1}{\sqrt{2\pi \Delta x^2(t)}} e^{-\frac{(x-\langle x(t)\rangle)^2}{2\Delta x^2(t)}}$$

$$x(t) = x_0 e^{-M_1 t} \qquad \Delta x^2(t) = \Delta x^2(0) e^{-2M_1 t} + \frac{M_2}{2M_1}(1-e^{-2M_1 t})$$

The Fokker-Planck equation maps Gaussian distributions to Gaussians. As a Gaussian is fully described by its mean and variance, $\langle x(t)\rangle$ and $\Delta x^2(t)$ describes the solution.

In Phase space, seperable in X & P

$$G(x-x', p-p', t) = \frac{1}{2\pi\sigma^2(t)} e^{-\frac{(x-x' e^{-M_1 t})^2}{2\sigma^2(t)}} e^{-\frac{(p-p' e^{-M_1 t})^2}{2\sigma^2(t)}} \qquad \sigma^2(t) = \frac{M_2}{2M_1}(1-e^{-2M_1 t})$$

For the case of the harmonic oscillator,

$$\frac{\partial W}{\partial t} = \{H_w, W\}_{MB} + \frac{\Gamma}{2}\left(\frac{\partial}{\partial x}(xW) + \frac{\partial}{\partial p}(pW)\right) + \frac{\Gamma}{2}(\bar{n}+\tfrac{1}{2})\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial p^2}\right)W$$

Drift coeffcient $M_1 = \frac{\Gamma}{2}$, Diffusion coefficient $M_2 = \frac{\Gamma}{2}(n+\frac{1}{2})$

$\Rightarrow$ In the absence of Hamiltonian evolution

$$\langle x\rangle(t) = \langle x\rangle(0)\, e^{-\Gamma t/2}, \quad \langle P\rangle(t) = \langle P\rangle(0)\, e^{-\Gamma t/2}$$

$$\langle \Delta X^2\rangle(t) = \langle \Delta X^2\rangle(0)\, e^{-\Gamma t} + (1-e^{-\Gamma t})(\bar{n}+\tfrac{1}{2})$$

$$\langle \Delta P^2\rangle(t) = \langle \Delta P^2\rangle(0)\, e^{-\Gamma t} + (1+e^{-\Gamma t})(\bar{n}+\tfrac{1}{2})$$

- The mean amplitude of X and P decay at $\frac{\Gamma}{2}$ to the origin
- The fluctuations in X and P come to a steady-state value of $\bar{n}+\frac{1}{2}$ the fluctuations of X and P associated with a thermal state $\rho = \frac{1}{z} e^{-\hat{n}\hbar\omega/kT}$

Note, at zero temperature $\bar{n}=0$, the steady-state (s.s.) values

$$\left.\begin{aligned} \langle X\rangle_{s.s.} &= \langle P\rangle_{s.s} = 0 \\ \langle \Delta X^2\rangle_{s.s.} &= \langle \Delta P^2\rangle_{s.s} = \tfrac{1}{2} \end{aligned}\right\} \text{ the steady-state is the vacuum}$$

Note, if the initial state is a coherent state, $|\alpha\rangle$, $\langle \Delta X^2\rangle(0) = \langle \Delta P^2\rangle(0) = \frac{1}{2}$

$\Rightarrow$ At zero temperature, $\bar{n}=0$, $\langle \Delta x^2\rangle(t) = \langle \Delta P^2\rangle(t) = \frac{1}{2}$ at all times

$\Rightarrow$ the state is a <u>coherent state at all times</u>.

We can see this by look at the solution to the Fokker-Planck equation

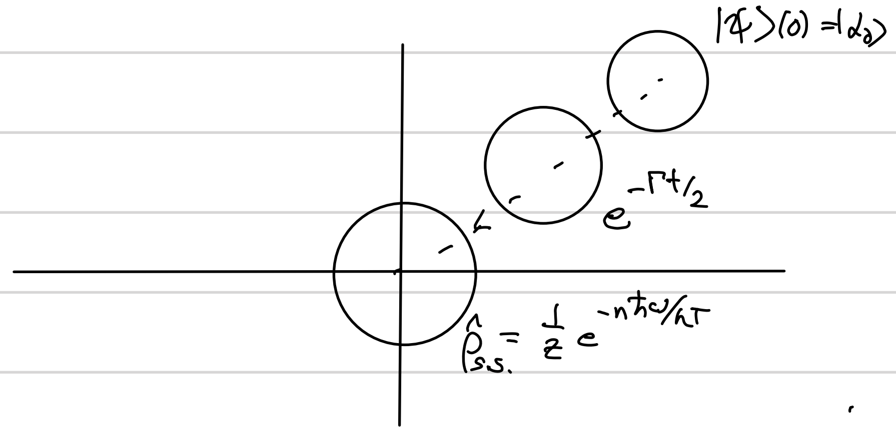
Given $|\psi(0)\rangle = |\alpha_0\rangle$, the initial Wigner function is

$$W(x,P,0) = \frac{1}{\pi} e^{-(x-x_0)^2} e^{-(P-P_0)^2} \quad \left(\begin{array}{l}\text{Gaussian with variance } \Delta x_0^2 = \Delta P_0^2 = \tfrac{1}{2} \\ \langle x\rangle(0) = x_0, \quad \langle P\rangle(0) = P_0\end{array}\right)$$

$$\Rightarrow W(x,P,t) = \frac{1}{\pi} e^{\frac{-(x-x_0e^{-\Gamma t/2})^2}{2\Delta x^2(t)}} e^{\frac{-(P-P_0e^{-\Gamma t/2})}{2\Delta P^2(t)}}$$

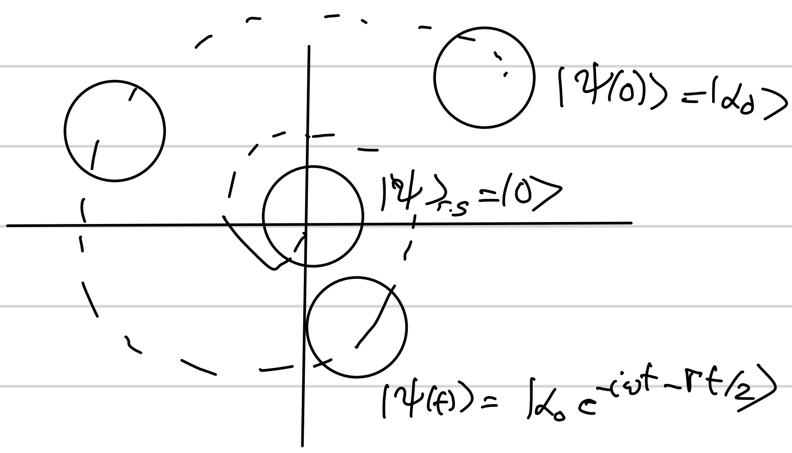
where $\Delta x^2(t) = \Delta P^2(t) = \frac{1}{2}e^{-\Gamma t} + (1-e^{-\Gamma t})(\bar{n}+\frac{1}{2}) = \frac{1}{2} + (1-e^{-\Gamma t})\bar{n}$

The phase space dynamics (in the rotating frame)



The state remains Gaussian, and spreads to a thermal state is steady state. At zero temperature, the state remains a coherent state at all times, but its <u>mean value decays along the classical trajectory</u>. In the Lab frame

$$|\psi(t)\rangle = |\alpha_0 e^{-i\omega t - \Gamma t/2}\rangle$$



The fact that the state remain pure under <u>open systems</u> dynamics is surprising. It is another unique property of the SHO. It arises because $|\alpha\rangle$ is an <u>eigenstate of the Lindblad operator</u> $\hat{L} = \sqrt{\Gamma}\,\hat{a}$. We will study this in more detail in homework. The implication is that coherent states are the most robust states the decoherence under the usual Lindblad equation. They are the so-called <u>"pointer states"</u> as we will study in more detail.

The role of quantum fluctuation

Classically, damping is most associated with dissipation of energy into the environment, in the form of heat. This dissipation leads to decay of oscillatory motion. There are also fluctuations, according to the fluctuation/dissipation theorem we will study in more detail, but these are less important.

Quantumly, the fluctuations play a hugely important role. Without quantum fluctuations from the environment, dissipation would damp quantum uncertainty an ultimately lead to a violation of the uncertainty principle. The feeding of quantum fluctuations follows from the diffusion term in the Fokker-Planck equation. This is also important is explaining decoherence, the process by which "nonclassicality" in a quantum state is lost due to its coupling to the environment.

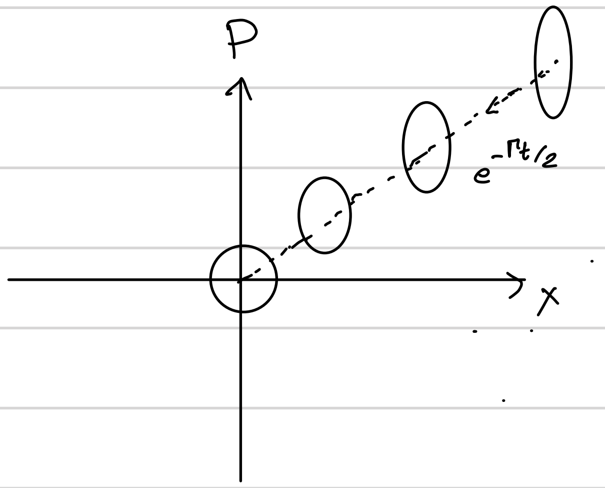
As a first simple example of how decoherence leads to a loss of nonclassicality, consider a squeezed state, with $\Delta X^2 = \frac{1}{2s}$, $\Delta P^2 = \frac{s}{2}$, $s > 1$, $\Delta X \Delta P = \frac{1}{2}$. The squeezed state is nonclassical in the sense that in one quadrature (here X), the fluctuations are smaller than that of the vacuum $\Delta X < \frac{1}{2}$. While nonclassical in this sense, the Wigner function is positive and thus Gaussian (the only pure state which is positive is Gaussian). Thus, we can simply write the solution to the master equation: At zero temperature, in the rotating

$$W(X,P,t) = \frac{1}{\pi \sigma_x(t)\, \sigma_y(t)} \exp\left\{-\frac{(X-\langle x\rangle(t))^2}{2\sigma_x^2(t)}\right\} \exp\left\{-\frac{(P-\langle P\rangle(t))^2}{2\sigma_p^2(t)}\right\}$$

$$\langle X\rangle(t) = \langle x\rangle(0)\, e^{-\Gamma t/2}, \quad \langle P\rangle(t) = \langle P\rangle(0)\, e^{-\Gamma t},$$

$$\sigma_x^2(t) = \frac{1}{2s} e^{-\Gamma t} + \frac{1}{2}(1-e^{-\Gamma t}), \qquad \sigma_p^2(t) = \frac{s}{2} e^{-\Gamma t} + \frac{1}{2}(1-e^{-\Gamma t})$$

For $0 < t < \infty$, the state is mixed. It is Gaussian state with $\Delta X \Delta P > \frac{1}{2}$. As $t \to \infty$, the state decays to the vacuum. Interestingly, the state loses purity as it decoherences, but ultimately decay to the vacuum (another pure state). This damped SHO at zero temperature is not a unital channel.



Decay of a squeezed state

$$\Delta X(t) = \frac{1}{2}\left(1 - \frac{s-1}{s} e^{-\Gamma t}\right) \qquad s>1$$

<u>Decoherence and the Quantum-to-Classical Transition</u>

Why do we not observe the effect of quantum interference of macroscopic distinguishable states, e.g., why do we not observable "Schrödinger's cat?" The answer is decoherence — the coupling to the environment causes the loss of coherence between distinguishable alternatives. This is a natural consequence of open quantum systems, as we studied when defining quantum channels. The environment has a record of different possible alternatives in the quantum state. But if we don't have access to that record we must trace out the environment. The result, generally, is a mixed state, and the loss of coherence.

Consider an initial "cat state" $|\psi(0)\rangle = \sqrt{N}\left(|\alpha_0\rangle + |-\alpha_0\rangle\right)$ where $N = \frac{1}{2(1+\mathrm{Re}\langle\alpha|-\alpha\rangle)}$

When $|\alpha_0| \gg 1$, this represents the superposition of two distinguishable alternatives $|\alpha_0\rangle$ "alive" and $|-\alpha_0\rangle$ "dead"

$$\hat{\rho}(0) = N\left(|\alpha_0\rangle\langle\alpha_0| + |-\alpha_0\rangle\langle-\alpha_0| + |\alpha_0\rangle\langle-\alpha_0| + |\alpha_0\rangle\langle-\alpha_0|\right)$$

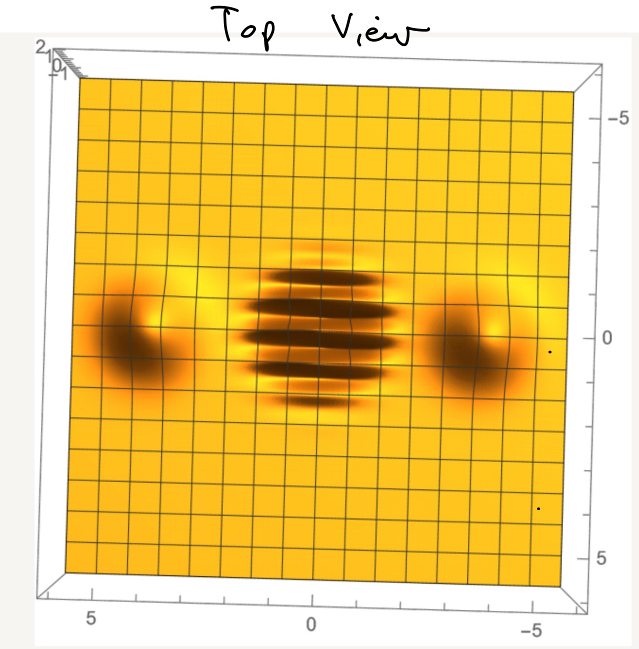
We see both the statistical mixture of the two alternatives and the coherences. The Wigner function at the initial time exhibits interference of negativity. Let us take $\alpha_0 = \frac{X_0}{\sqrt{2}}$ (Real). As we found in homework

$$\Rightarrow W(X,P,0) = N\left[W_+(X,P) + W_-(X,P) + 2\,\frac{e^{-X^2+P^2}}{\pi}\cos(2X_0 P)\right]$$

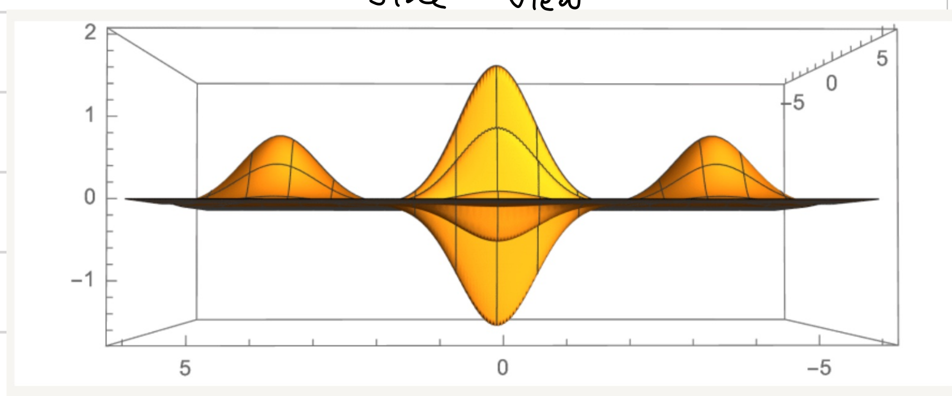
$$W_\pm(X,P) = \frac{2}{\pi}\, e^{-(X\mp X_0)^2 + P^2}$$

are the Wigner functions of the coherent state localized at $\pm X_0$

Top View



Side View



Cat state with $X_0 = 4$ $\alpha = \frac{X_0 + iP_0}{\sqrt{2}}$

Let us consider how such a state will evolve under the master equation of a simple harmonic oscillator. In particular, in the phase space representation, how does the cat state evolve under the Fokker-Planck equation?

$$\frac{\partial W}{\partial t} = \frac{\Gamma}{2}\left(\frac{\partial}{\partial x}(xW) + \frac{\partial}{\partial p}(pW)\right) + \frac{\Gamma}{2}(\bar{n}+\tfrac{1}{2})\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial p^2}\right) W$$

In particular, when $X_0 \gg 1$, approaching something macroscopic, there are very different effects of the drift and diffusion terms.

As we saw, the drift term will cause each Gaussian wavepacket (coherent state) to decay towards the original at rate Γ. This is the effect of energy dissipation. The diffusion term will also act on each coherent state to thermalize the fluctuations, according to fluctuation-dissipation. This also occurs at the rate Γ.

However, importantly, the effect of diffusion on the interference terms is markedly different. When $X_0 \gg 1$,

$$\frac{\Gamma}{2}(\bar{n}+\tfrac{1}{2})\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial p^2}\right) W \approx \frac{\Gamma}{2}(\bar{n}+\tfrac{1}{2})\left(\frac{2}{\pi} e^{-(x^2+p^2)}\right)\frac{\partial^2}{\partial p^2}\cos(2X_0 p) = -2\Gamma(\bar{n}+\tfrac{1}{2})X_0^2 \, \frac{2}{\pi} e^{-(x^2+p^2)} \cos(2X_0 p)$$

That interference terms decay at the rate $2\Gamma(\bar{n}+\frac{1}{2})X_0^2 \gg \Gamma$

Consider the high temperature limit $\bar{n} \approx \frac{kT}{\hbar\omega} \gg 1$, and the case of a massive particle is a harmonic well, so $X_0 = \frac{x_0}{\sqrt{\hbar/m\omega}} \Rightarrow X_0^2 = x_0^2 \frac{m\omega}{\hbar}$

$$\Rightarrow \Gamma_{diffusion} = 2\Gamma(\bar{n}+\tfrac{1}{2})X_0^2 \approx \Gamma\left(\frac{2mkT}{\hbar^2}\right)x_0^2 = \Gamma\left(\frac{x_0}{\bar{\lambda}_{dB}}\right)^2$$

where $\bar{\lambda}_{dB} = \frac{\lambda_{dB}}{2\pi}$, $\lambda_{dB} = \frac{h}{\sqrt{2mk_BT}}$ is the de Broglie wavelength of the particle a temperature T.

As a concrete example, less consider an extreme "macroscopic case"

- Take $x_0 = 1$ cm, $m = 1$ gram, $T = 300K$ (room temperature)

$\Rightarrow \frac{\Gamma_{diffusion}}{\Gamma} \approx 10^{40}$ Woah!

Even if $\tau_{decay} = \frac{1}{\Gamma} = 10^{17}$ s (age of the universe)

$\tau_{diffusion} \approx 10^{-23}$ s, essentially instantaneous

The diffusion term in the Fokker–Planck equation dominates the effect of decoherence. The nonclassical features are associated with rapid oscillation of the Wigner function. This fine-grained structure has the highest curvature, and thus the fastest rate of diffusion which is proportional to $(\partial_x^2+\partial_p^2)W$. In the case of the cat state, the interference terms diffuse away much more rapidly than energy decays when $X_0 \gg 1$, and the more macroscoping the cat, the finer grained the structure, and thus the faster the decoherence. The more nonclassical the state the more "fragile" it is!

The environment monitors the state of the system. For this massive particle, when the environment can distinguish the position of the particle, the coherence between different positions is lost. In the high temperature limit the scale of distinguishability is the particles de Broglie wavelength. Even at zero temperature decoherence will occur, due to quantum fluctuations. For $\bar{n}=0$

$$\Gamma_{diffusion} = \Gamma_{decoherence} = \Gamma\left(\frac{x_0}{x_{vac}}\right)^2 \quad \text{where} \quad X_{vac} = \sqrt{\frac{\hbar}{2m\omega}} = \text{zero point motion.}$$

Another way of understanding this is to consider the number of quanta

$|\alpha_0|^2 = \frac{X_0^2}{2} = \langle n_0\rangle$: Mean number of quanta in the coherent state

$\Rightarrow$ At zero temperature $\Gamma_{decoherence} = \Gamma X_0^2 = 2\Gamma\langle n_0\rangle$.

For a macroscopic system, $\langle n_0\rangle \gg 1$, and decoherence happens very rapidly. This is true for a general bosonic field.

From the point of view of quanta (e.g. photons), leaking even <u>one photon</u> into the environment can distinguish $|\alpha_0\rangle$ from $|-\alpha_0\rangle$. To see this, let us return to our model of the couplin of the system and environment. Suppose the bath is initial in the vacuum, and the system is in the coherent state $|\alpha_0\rangle$. Then after a short time δt, the joint evolution is defined by the quantum channel. For time δt

$$\hat{U}_{int}|\alpha_0\rangle|0\rangle = M_0(\delta t)|\alpha_0\rangle|0\rangle + M_1(\delta t)|\alpha_0\rangle|1\rangle \leftarrow \text{One photon in the bath}$$

where $\hat{M}_0(\delta t) = \hat{\mathbb{1}} - \frac{\hat{L}^\dagger\hat{L}}{2}\delta t$, $\hat{M}_1(\delta t) = \hat{L}\sqrt{\delta t}$ $\hat{L} = \sqrt{\Gamma}\,\hat{a}$

$$\Rightarrow \hat{U}_{int}|\alpha_0\rangle|0\rangle = \left(\hat{\mathbb{1}} - \frac{\Gamma\delta t}{2}\hat{a}^\dagger\hat{a}\right)|\alpha_0\rangle|0\rangle + \sqrt{\Gamma\delta t}\,\hat{a}|\alpha_0\rangle|1\rangle$$

Aside $\left(\hat{\mathbb{1}} - \frac{\Gamma\delta t}{2}\hat{a}^\dagger\hat{a}\right)|\alpha_0\rangle = \sum_n e^{-|\alpha_0|^2/2}\left(1-\frac{\Gamma\delta t}{2}\right)^n \frac{\alpha^n}{\sqrt{n!}}|n\rangle \underset{\text{order } \delta t}{\approx} \sum_n e^{-|\alpha_0|^2/2}\frac{(\alpha e^{-\frac{\Gamma\delta t}{2}})^n}{\sqrt{n!}}|n\rangle$

$$= e^{-|\alpha_0|^2/2} e^{+\frac{|\alpha_0 e^{-\frac{\Gamma\delta t}{2}}|^2}{2}}|\alpha_0 e^{-\Gamma t/2}\rangle \approx e^{-\frac{|\alpha_0|^2\Gamma\delta t}{2}}|\alpha_0\rangle \quad (\Gamma\delta t \ll 1)$$

$$\approx \left(1 - |\alpha|_0^2\frac{\Gamma}{2}\delta t\right)|\alpha_0\rangle \quad (|\alpha_0|^2\Gamma\delta t \ll 1)$$

$$\Rightarrow \hat{U}_{int}|\alpha_0\rangle|0\rangle \approx |\alpha_0\rangle\left[\left(1 - \frac{\Gamma}{2}|\alpha_0|^2\delta t\right)|0\rangle + \alpha_0\sqrt{\Gamma\delta t}\,|1\rangle\right]$$

Similarly if $|\psi(0)\rangle = |-\alpha_0\rangle$

$$\hat{U}_{int}|-\alpha_0\rangle|0\rangle \approx |\alpha_0\rangle\left(\left(1-\frac{\Gamma}{2}|\alpha_0|^2\delta t\right)|0\rangle - \alpha_0\sqrt{\Gamma\delta t}\,|1\rangle\right]$$

-

The environment has a record of the state of the system through the emission of a photon. This record distinguishes $|\alpha_0\rangle$ from $|-\alpha_0\rangle$ when these records are orthogonal

The overlap of the two meter states

$$\left[\left(1-\frac{\Gamma}{2}|\alpha_0|^2\delta t\right)\langle 0| - \alpha_0^*\sqrt{\Gamma\delta t}\,\langle 1|\right]\left[\left(1-\frac{\Gamma}{2}|\alpha_0|^2\delta t\right)|0\rangle + \alpha_0\sqrt{\Gamma\delta t}\,|1\rangle\right]$$

$$= 1 - 2\Gamma|\alpha_0|^2\,\delta t \quad \text{(to order } \delta t)$$

$\Rightarrow$ when $2\Gamma|\alpha_0|^2\delta t \approx 1$, the environment distinguishes $|\alpha_0\rangle$ from $|-\alpha_0\rangle$ just as we saw from the Fokker-Planck equation.

The information about the phase of the coherent state is stored in the phase of the photon. We can measure that using a "homodyne detector," well known in quantum optics. The degree to which we can resolve this phase is limited by the "shot noise" in the detector, which is fundamentally determined by vacuum noise.

For two general coherent states $|\alpha_0\rangle$ and $|\beta_0\rangle$ the overlap

$$\left[\left(1-\frac{\Gamma}{2}|\alpha_0|^2\delta t\right)\langle 0| + \alpha_0^*\sqrt{\Gamma\delta t}\,\langle 1|\right]\left[\left(1-\frac{\Gamma}{2}|\beta_0|^2\delta t\right)|0\rangle + \beta_0\sqrt{\Gamma\delta t}\,|1\rangle\right.$$

$$\Rightarrow 1 - \frac{\Gamma}{2}\left(|\alpha_0|^2+|\beta_0|^2-2\alpha_0^*\beta\right)\delta t \approx \langle\alpha_0|\beta_0\rangle^{-\Gamma\delta t} \quad \text{to order } \delta t$$

Thus, the degree to which the meter can distinguish the two coherent states is determined by their overlap.

<u>Decoherence and the emergence of the classical world</u>

Classical and quantum mechanics differ in a number of ways. One way is dynamics. The expected value of $\langle\hat{x}\rangle$ and $\langle\hat{p}\rangle$ quantumly do not generally follow the classically predicted values even for a minimum uncertainty wavepacket. We saw this in Lecture 6. That for "nonlinear" motion (beyond a harmonic potential), $\left\langle \frac{\partial V}{\partial x}\Big|_{x=\hat{x}}\right\rangle \neq \frac{\partial V(x)}{\partial x}\Big|_{x=\langle\hat{x}\rangle}$

When a wavepacket spreads over a scale $\chi \equiv \sqrt{\frac{\partial_x V}{\partial_x^3 V}}$

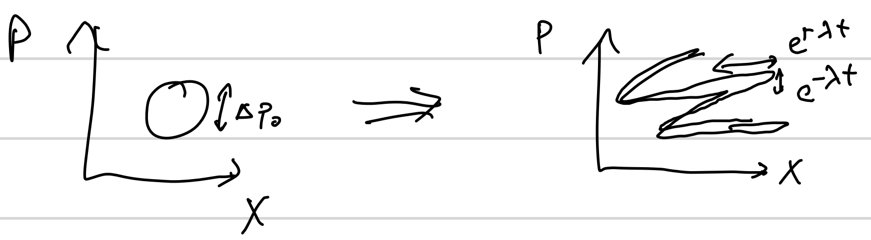
(The scale of the force compared to the first correction), the $\langle\hat{x}\rangle$ will diverge from the classical trajectory. More quantitatively, we saw that the equation of motion for the Wigner function under Hamiltonian dynamics is for a particle in a well

$$\frac{\partial W}{\partial t} = \underset{\substack{\uparrow \\ \text{Moyal Bracket}}}{\{H, W\}_{MB}} = \underset{\substack{\uparrow \\ \text{Poisson Bracket}}}{\{H, W\}_{PB}} + \sum_{n=1}^{\infty} \frac{\hbar^{2n} (-1)^n}{2^{2n} (2n+1)!} \partial_x^{2n+1} V \, \partial_p^{2n+1} W(x,p)$$

The first term represents the classical flow according to the Liouville equation. The second term represents the quantum corrections. The time scale in which these terms kick in for a well-localized wavepacket is know as the Ehrenfest time. For a macroscopic system, when X is macrosopic, it is often taught that as $\hbar \to 0$, the Ehrenfest time $\to \infty$, so in the "correspondence principle limit" the dynamics is classical for macroscopic systems.

However, this is naive. In classically chaotic dynamics, nearby trajectories can exponentially diverge. The classical flow will thus cause the wavepacket to spread exponentially fast, and the Ehrenfest time $\tau_E \sim \frac{1}{\lambda} \log\left(\frac{I}{\hbar}\right)$, where λ is the rate of exponential spreading (the Lyapunov exponent) and I is the characteristic "action" associated with the classical orbit. The logarithmic dependence on $I/\hbar$, means even for macroscopic objects, the Ehrenfest time will be relatively short and quantum corrections to dynamics will become important.

From the equation of motion for the Wigner function, we see that quantum corrections to classical dynamics arise from the fine grained structure. As chaotic dynamics stretch & fold an initial patch of phase space, fine grained structure arises



We expect as Δp becomes narrow $\Delta p(t) \sim \Delta p_0 e^{-\lambda t}$ we will reach the quantum regime exponential fast. For $\Delta X(t) \sim \Delta x_0 e^{+\lambda t} = \frac{\hbar}{\Delta p_0} e^{\lambda t}$, the Ehrenfest time $\frac{\hbar}{\Delta p_0} e^{\lambda t_E} = X \Rightarrow t_E = \frac{1}{\lambda} \log\left(\frac{\Delta p_0 X}{\hbar}\right)$.

The paradoxical behavior was pushed to its limits in a jarring example put forward by Zurek (Physica Scripta 76, 186 (1998)). Hyperion, one of the moons of Saturn, shaped like an elongated ellipse, is known to have a chaotic orbit on a scale $\lambda^{-1} = 2T$, $T = 21$ day = periodic of orbit. The logarithmic dependence on $\hbar^{-1}$, means the Ehrenfest time will be relatively short. Taking $\Delta p_0 = \frac{\hbar}{\lambda_{\text{deBroglie}}}$ is space, Zurek found

$$\tau_E \approx 20 \text{ yr } !$$

Of course we do expect the trajectory of a moon to be governed by quantum dynamics. The resolution to this is that Hyperion is an open quantum system, constantly monitored by it environment. Decoherence limit the extent of quantum coherence of a wavepacket, which would cause as to predict a different trajectory than we would expect classically.

More quantitatively, the master equation includes the Lindbladian

$$\frac{\partial W}{\partial t} = \mathcal{L}[W] = \Gamma\left(\frac{\partial}{\partial x}(xW) + \frac{\partial}{\partial p}(pW)\right) + \frac{\Gamma}{2}(\bar{n}+\tfrac{1}{2})\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial p^2}\right)W$$

In paricular, the diffusion term, responsible for decoherence, counteract the "squeezing effect" and limit the fine-grained structure that can form thus recovering the classical behavior. To estimate this balance between squeezing a decoherence set

$$\tau_D^{-1} \approx \lambda \quad , \text{ where } \tau_D^{-1} = \Gamma\left(\frac{l_c}{\Lambda_{dB}}\right)^2 \quad , \; l_c = \text{coherence length}$$

$$\Rightarrow l_c = \Lambda_{dB}\sqrt{\frac{\lambda}{\Gamma}} \quad \text{(Wave packet has limited coherence).}$$

Now, as Λ_{dB} is tiny, even at temperature of space ($\to$ zero point motion at zero temperature)

$$l_c <<<< X \quad \text{for macroscopic systems}$$

$\Rightarrow$ Decoherence happens essentially instantaneously compared to Ehrenfest time expected by closed quantum system dynamics. Decoherence explains the quantum-to-classical transition.

For sufficient macroscopic systems, the diffusion term makes the quantum corrections to the Poisson bracket small in which case

$$\frac{\partial W}{\partial t} = \{H, W\}_{PM} + \mathcal{L}[W]$$

This is the truncated Wigner approximation (TWA), including the Linbladian. Even with very weak coupling to the environment, for macroscopic systems, the smallest amount of decoherence can kill nonclassical dynamics, so the dynamics are very close to the Hamiltonian classical dynamics.

An example of this was studied in "Decoherence, Chaos, and the Correspondence Principle" by S. Habib, K. Shizume, and W. Zurek PRL $\underline{80}$ 4361 (1993), the example of the "Duffing oscillator" with diffusion

$$\hat{H} = \frac{\hat{p}^2}{2m} + Bx^4 - Ax^2 + Cx\cos(\omega t) \leftarrow \text{Oscillation of double-well}$$

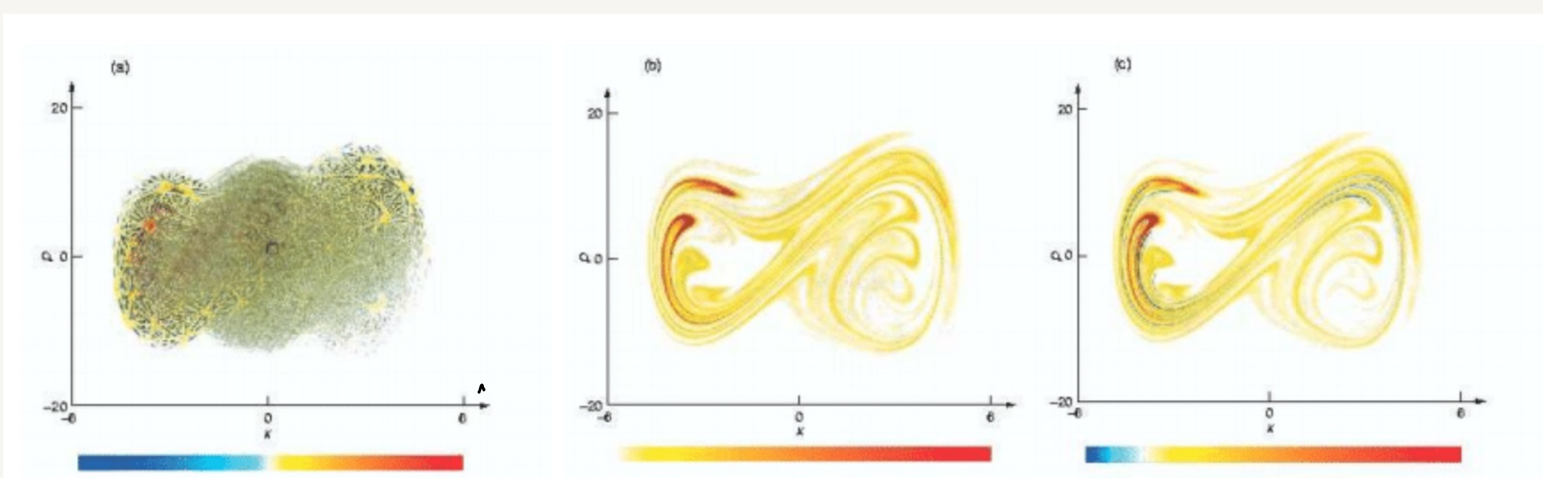
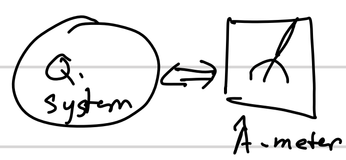


Figure B: Decoherence in a Chaotic Driven Double-Well System. This numerical study (Habib et al. 1998) of a chaotic driven double-well system described by the Hamiltonian $H = p^2/2m - Ax^{+}Bx^4 + Fx\cos(\omega t)$ with $m = 1, A = 10, B = 0.5, F = 10$, and $\omega = 6.07$ illustrates the effectiveness of decoherence in the transition from quantum to classical. These parameters result in a chaotic classical system with a Lyapunov exponent $\lambda \cong 0.5$. The three snapshots taken after 8 periods of the driving force illustrate phase space distributions in (a) the quantum case, (b) the classical case, and (c) the quantum case but with decoherence ($D = 0.025$). The initial condition was always the same Gaussian, and in the quantum cases, the state was pure. Interference fringes are clearly visible in (a), which bears only a vague resemblance to the classical distribution in (b). By contrast, (c) shows that even modest decoherence helps restore the quantum-classical correspondence. In this example the coherence length ℓ_C is not much smaller than the typical nonlinearity scale, so the system is on the border between quantum and classical. Indeed, traces of quantum interference are still visible in (c) as blue "troughs," or regions where the Wigner function is still slightly negative. The change in color from red to blue shown in the legends for (a) and (c) corresponds to a change from positive peaks to negative troughs. In the ab initio classical case (b), there are no negative troughs.

Decoherence and the "measurement problem"

In the foundational Copenhagen formulation of quantum mechanics measurement plays a special role as the measuring apparatus in "classical" outside the quantum realm. A measurement in said to cause a "collapse of the wavefunction" into an eigenstate of the observable being measured. This is, of course, incredibly unsatisfying and has caused controversy and debate every since. The apparatus is physical stuff and should be fundamentally governed by quantum mechanics. So how does a measurement "happen" and which kind of physical interactions correspond to a measurement.

In his foundational paper, Von Neumann gave a (now partial) answer, in which the measurement was treated quantum mechanics. As we studied in homework, we can consider a quantum meter coupled to a quantum system



In a QND measurement $\hat{H}_{int} = \chi \hat{A}_s \otimes \hat{P}_M$, the meter is prepared in $|0\rangle_M$

then $\hat{U}_{int} |\psi\rangle_s \otimes |0\rangle_M = e^{-i\chi t \hat{A}_s \otimes \hat{P}_M} \sum_a c_a |a\rangle_s \otimes |\Phi_a\rangle_M$ where $|\Phi_a\rangle_m = e^{-i\chi a \hat{P}_M} |0\rangle_M$

The record of the a eigenvalue a is correlated with the meter "ponter state" $|\Phi_a\rangle_m$. So if $\{|\Phi_a\rangle_m\}$ are distinguishable, when we "find" the pointer a $|\Phi_a\rangle_m$, we learn thee system is in $|a\rangle_s$. But of course, this doesn't fully solve the measurement problem because we have to determine the state of meter by measuring it.

For in principle we could to do measurement the find the meter in the state $\alpha_1 |\Phi_{a_1}\rangle + \alpha_2 |\Phi_{a_2}\rangle$ and then learn that the state is in $\alpha_1 |a_1\rangle_s + \alpha_2 |a_2\rangle_s$. So the apparatus, as it stands, does not specify the basis in which we measured the system.

A resolution to this was proposed by Wojciech Zurek in 1981 (which gained traction in the 1990's. Critically the system and meter are not isolated from the environment. It is an open quantum system. Importantly the meter itself is meant to by a macroscopic system — this was Bohr's notion of the classical apparatus. And as we have seen, macroscopic superpositions are fragile in the face of decoherence. The enviroment does determines the basis in which the measurement is done.

More formally, we now include the environment as a third subsystem.

$$U_{SME} |\psi\rangle_S \otimes |0\rangle_M \otimes |0\rangle_E = \sum_a c_a |a\rangle_S \otimes |\Phi_a\rangle_M \otimes |\chi_a\rangle_E$$

This form depends on nature of meter ~ environment interaction

Assuming ${}_E\langle \chi_a | \chi_{a'} \rangle = \delta_{aa'}$, then tracing out the environment

$$\hat{\rho}_{SM} = \sum_a |c_a|^2 |a\rangle_S\langle a| \otimes |\Phi_a\rangle_M\langle \Phi_a|$$

The system and meter are now classically correlated: The density operator is diagonal in the basis that determines the measurement. Importantly, the enviroment determines the basis in which the system-meter decoheres

The basis in which the density matrix is diagonal is known as the "pointer basis" in its relation to the von Neumann measurement paradigm. The pointer basis is robust to decoherence. It is determined by its interaction which the environment. As the pointer is a macroscopic object, by definition, due to decoherence, it will not be in a macroscopic superposition.

Beyond the von Neumann measurement paradigm, the notion of the "pointer basis" general describes the basis in which the system will decoheres in the presence of infraction with the environment. In a Markov system, it in determined by the Lindbland operators. For example for the damped SHO, the Lindblad operator in $\hat{L} = \sqrt{\Gamma}\,\hat{a}$. Thus the pointer states are <u>coherent states</u> $|\alpha\rangle$. These are the "classical states" because they are the most robust to decoherence.