Phys 581 - Open Quantum Systems

Lecture 14: The Stochastic Schrödinger Equation: Quantum State Diffusions

Formal description of Jump (Poisson) Process

We have seen that the Lindblad master equation

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}\left[\hat{H}_{eff}, \hat{\rho}\right] + \sum_{\mu} \hat{L}_\mu \hat{\rho} \hat{L}_\mu^\dagger , \quad \text{where} \quad \hat{H}_{eff} = \hat{H} - \frac{i\hbar}{2}\sum_\mu \hat{L}_\mu^\dagger \hat{L}_\mu$$

is formal equivalent to the ensemble average over all quantum trajectories

$$\hat{\rho}(t) = \langle\langle |\psi(t)\rangle\langle\psi(t)| \rangle\rangle$$

Where $|\psi(t)\rangle$ evolves stochastically according to

$$|\psi(t+dt)\rangle \Rightarrow \begin{cases} \dfrac{\hat{L}_\mu|\psi\rangle}{\|\hat{L}_\mu|\psi\rangle\|} & \text{with probability } dp_\mu(t) = \langle\psi(t)|\hat{L}_\mu^\dagger \hat{L}_\mu|\psi(t)\rangle dt \\ \dfrac{(\mathbb{1} - \frac{i}{\hbar}\hat{H}_{eff}\, dt)|\psi(t)\rangle}{\|(\mathbb{1} - \frac{i}{\hbar} H_{eff}\, dt)|\psi(t)\rangle\|} & \text{with probability } p_0(t) = 1 - \sum_\mu dp_\mu(t) \end{cases}$$

We can formally express the evolution of $|\psi(t)\rangle$ as a "stochastic differential equation"

$$|\psi(t+dt)\rangle = \left(1 - \sum_\mu dN_\mu\right)\frac{(\mathbb{1} - \frac{i}{\hbar}\hat{H}_{eff}\, dt)|\psi(t)\rangle}{\|(\mathbb{1} - \frac{i}{\hbar} H_{eff}\, dt)|\psi(t)\rangle\|} + \sum_\mu dN_\mu(t)\, \frac{\hat{L}_\mu|\psi\rangle}{\|\hat{L}_\mu|\psi\rangle\|}$$

where $dN_\mu(t)$ is a "stochastic interval", taking on random values

$$dN_\mu(t) = \begin{cases} 1 & \text{with probability } dp_\mu(t) = \langle\psi(t)|\hat{L}_\mu^\dagger \hat{L}_\mu|\psi(t)\rangle dt \\ 0 & \text{with probability } 1 - dp_\mu(t) \end{cases}$$

the intervals at different times in uncorrelated, making this a "Poisson process" associated with counting statistics. From these it follows

$$\langle\langle dN_\mu \rangle\rangle = dp_\mu = \langle\psi(t)|\hat{L}_\mu^\dagger \hat{L}_\mu|\psi(t)\rangle\, dt$$

$$dN_\mu\, dN_\nu = dN_\mu\, \delta_{\nu\mu}$$

To rule of stochastic calculus is then to keep term to $O(dt)$. Since $dN = O(dt)$
$dN\,dt = 0$

The stochastic equation is then

$$|\psi(t+dt)\rangle = \frac{\left(1 - \frac{i}{\hbar}dt\,\hat{H}_{eff}\right)|\psi(t)\rangle}{\sqrt{\underbrace{\langle\psi(t)|\,1 - \frac{i}{\hbar}\left(\hat{H}_{eff} - \hat{H}_{eff}^{\dagger}\right)dt\,|\psi(t)\rangle}_{\substack{\shortparallel \\ 1 - \sum_\mu \langle\psi|\hat{L}_\mu^\dagger \hat{L}_\mu|\psi\rangle dt}}}} + \sum_\mu{}' dN_\mu \left[\frac{\hat{L}_\mu}{\sqrt{\langle\psi(t)|\,\hat{L}_\mu^\dagger\hat{L}_\mu|\psi(t)\rangle}} - 1\right]|\psi(t)\rangle$$

$$= \left(1 - \frac{i}{\hbar}\hat{H}_{eff}\,dt + \frac{1}{2}\sum_\mu \langle\hat{L}_\mu^\dagger\hat{L}_\mu\rangle dt\right)|\psi(t)\rangle + \sum_\mu dN_\mu \left[\frac{\hat{L}_\mu}{\sqrt{\langle\hat{L}_\mu^\dagger\hat{L}_\mu\rangle}} - 1\right]|\psi(t)\rangle$$

$$\Rightarrow \boxed{\begin{aligned} |d\psi(t)\rangle &= |\psi(t+dt)\rangle - |\psi(t)\rangle \\ &= \left(-\frac{i}{\hbar}\hat{H}_{eff} + \frac{1}{2}\sum_\mu \langle\hat{L}_\mu^\dagger\hat{L}_\mu\rangle\right) dt\,|\psi\rangle + \sum_\mu \left[\frac{\hat{L}_\mu}{\sqrt{\langle\hat{L}_\mu^\dagger\hat{L}_\mu\rangle}} - 1\right] dN_\mu\,|\psi\rangle \end{aligned}}$$

This is a form of the Stochastic Schrödinger Equation (SSE). It is stochastic because of the stochastic interval dN_μ. It is also nonlinear, as expected, through the renormalization. Note we can write this formally as

$$\frac{d}{dt}|\psi(t)\rangle = \left(-\frac{i}{\hbar}\hat{H}_{eff} + \frac{1}{2}\sum_\mu\langle\hat{L}_\mu^\dagger\hat{L}_\mu\rangle\right)|\psi(t)\rangle + \sum_\mu \left[\frac{\hat{L}_\mu}{\sqrt{\langle\hat{L}_\mu^\dagger\hat{L}_\mu\rangle}} - 1\right]\frac{dN_\mu}{dt}|\psi(t)\rangle$$

where $\frac{dN_\mu}{dt}$ = Current of random counts.

We show that the SSE, when ensemble averaged, yields the Lindblad Master Equation:

$$d\rho = \langle\langle d\,|\psi\rangle\langle\psi|\,\rangle\rangle = \langle\langle\, |\psi\rangle\langle d\psi| + |d\psi\rangle\langle\psi| + \underbrace{|d\psi\rangle\langle d\psi|}_{\substack{\shortparallel \\ \text{In the stochastic calculus, we must} \\ \text{be careful to keep terms of order } dt}}\,\rangle\rangle$$

$$\Rightarrow d\hat\rho = \langle\langle -\frac{i}{\hbar} H_{eff}\, dt\, |\psi\rangle\langle\psi| + \frac{i}{\hbar} |\psi\rangle\langle\psi| \hat H_{eff}^\dagger dt + \sum_\mu \langle \hat L_\mu^\dagger \hat L_\mu\rangle dt\, |\psi\rangle\langle\psi| \rangle\rangle$$

$$+ \sum_\mu \langle\langle dN_\mu \left(\frac{\hat L_\mu}{\sqrt{\langle \hat L_\mu^\dagger \hat L_\mu\rangle}} - 1\right) |\psi\rangle\langle\psi| + |\psi\rangle\langle\psi| \left(\frac{\hat L_\mu^\dagger}{\sqrt{\langle \hat L_\mu^\dagger \hat L_\mu\rangle}} - 1\right) \rangle\rangle$$

$$+ \sum_\mu \langle\langle dN_\mu \left(\frac{\hat L_\mu}{\sqrt{\langle \hat L_\mu^\dagger \hat L_\mu\rangle}} - 1\right) |\psi\rangle\langle\psi| \left(\frac{\hat L_\mu^\dagger}{\sqrt{\langle \hat L_\mu^\dagger \hat L_\mu\rangle}} - 1\right) \rangle\rangle$$

$$= -\frac{i}{\hbar} [\hat H_{eff}, \hat\rho]^\flat dt + \sum \langle \hat L_\mu^\dagger \hat L_\mu\rangle dt\, \hat\rho + \sum_\mu \langle\langle dN_\mu\rangle\rangle \left(\frac{\hat L_\mu \hat\rho \hat L_\mu^\dagger}{\sqrt{\langle \hat L_\mu^\dagger \hat L_\mu\rangle}} - \hat\rho \right)$$

Treating dN_μ and $|\psi\rangle\langle\psi|$ as statistically independent $\langle\langle dN_\mu |\psi\rangle\langle\psi|\rangle\rangle = \langle\langle dN_\mu\rangle\rangle \hat\rho$ (gives the right answer for the wrong reason – see D. Steck "Quantum Optics")

Now $\langle\langle dN_\mu\rangle\rangle = \langle \hat L_\mu^\dagger \hat L_\mu\rangle dt$

$$\Rightarrow \frac{d\hat\rho}{dt} = -\frac{i}{\hbar} [\hat H_{eff}, \hat\rho]' + \sum_\mu \hat L_\mu \hat\rho \hat L_\mu^\dagger \quad \checkmark \quad \text{q.e.d.}$$

<u>Weiner Process and Quantum State Diffusion</u>

We have seen that we obtain an equivalence class of "unravelings" of the master equation with a unitary remixing of the Krause operators $\hat M_\mu = \hat L_\mu \sqrt{dt}$, $\mu = 1, \dots, m$. This is a limited class of remixings. A more general equivalence is found including $\hat M_0 = \mathbb{1} - \frac{i}{\hbar} \hat H_{eff}\, dt$.

Consider a master equation with one Lindblad operator $\hat L$, and thus two Kraus operators $\hat M_1 = \hat L \sqrt{dt}$, $\hat M_0 = \mathbb{1} - \frac{i}{\hbar} \hat H dt - \frac{1}{2} \hat L^\dagger \hat L dt$. We can define a unitary remixing

$$\begin{bmatrix} \hat N_0 \\ \hat N_1 \end{bmatrix} = \begin{bmatrix} \mathbb{1} - \frac{1}{2}|A|^2 dt & -A^* \sqrt{dt} \\ A\sqrt{dt} & \mathbb{1} - \frac{1}{2}|A|^2 dt \end{bmatrix} \begin{bmatrix} \hat M_0 \\ \hat M_1 \end{bmatrix}$$

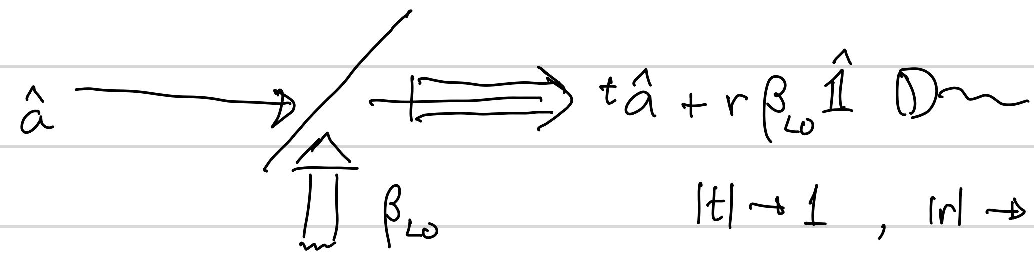
$$\Rightarrow \hat N_1 = \underbrace{(\hat L + A)}_{\equiv \hat J} \sqrt{dt}, \quad \hat N_0 = \mathbb{1} - \frac{1}{2}|A|^2 dt - \frac{i}{\hbar} \hat H dt - \frac{1}{2} \hat L^\dagger \hat L dt - A^* \hat L dt$$

$$= \mathbb{1} - \frac{i}{\hbar} \hat H dt - \frac{1}{2} (\hat L + A)^\dagger (\hat L + A)^{dt} + \frac{1}{2} (A \hat L^\dagger - A^* \hat L) dt$$

$$= \hat{\mathbb{1}} - \frac{i}{\hbar} \tilde H dt - \frac{1}{2} \hat J^\dagger \hat J dt$$

$$\boxed{\text{New jump operator } \hat J = \hat L + A, \quad \text{New Hamiltonian } \tilde H = \hat H - \frac{i\hbar}{2} (A^* \hat L - A \hat L^\dagger)}$$

We can always interpret a particular set of jump operators (and thus unravelling of the master equation) as associated with a particular measurement done on the environment. We recognize this jump operator as corresponding to unbalanced homodyne detection.

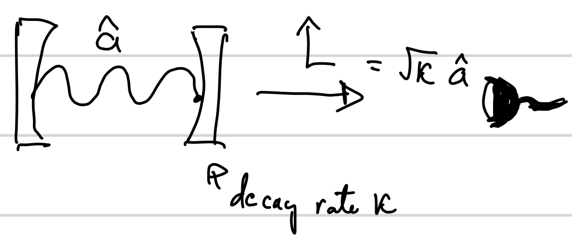


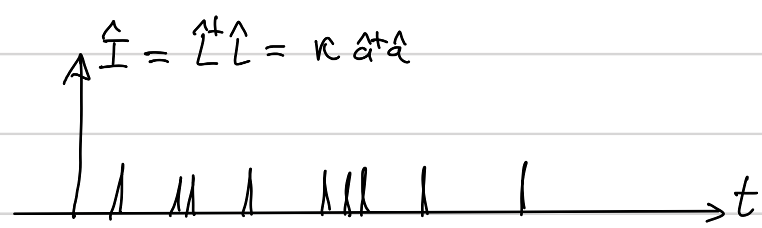
$|t| \to 1$, $|r| \to 0$, $|\beta_{LO}| \to \infty$

$r\beta_{LO} \to A$

For example: Decaying SHO

Direct Detection

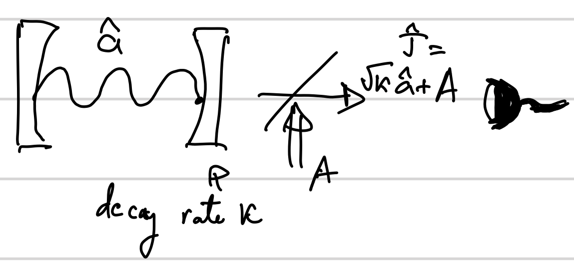


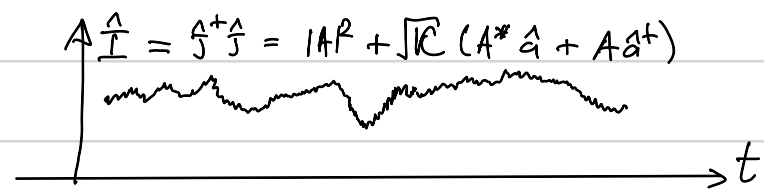


Poisson distributed counts

$\hat{L}^\dagger\hat{L} = \kappa\hat{a}^\dagger\hat{a}$ = Flux of output photons (rate)

Homodyne Detection





Continuous signal with noise

$|A|^2$ = Flux of local oscillator ; $|A| \to \infty$: Macroscopic # of photons in time $\frac{1}{\kappa}$

Define $A = \sqrt{\kappa}\,\alpha$, $|\alpha|^2 \gg 1$

In order to make the transition from the discrete Poisson process to continuous noise, we "coarse grain" the quantum trajectory. Consider a coarse-grained time interval δt

$$\frac{1}{|A|^2} = \frac{1}{\kappa|\alpha|^2} \ll \delta t \ll \frac{1}{\kappa}$$

In this time interval the $\langle\langle \delta N \rangle\rangle = \langle \hat{J}^\dagger \hat{J}\rangle \delta t = |A|^2\delta t + \langle A\hat{L}^\dagger + A^*\hat{L}\rangle \delta t + \langle \hat{L}^\dagger\hat{L}\rangle \delta t \gg 1$

In the limit of large mean, the Poisson distribution is well approximated by a Gaussian with fluctuations about mean $\langle\langle(\delta N - \langle\langle \delta N\rangle\rangle)^2\rangle\rangle = \langle\langle \delta N \rangle\rangle$. This can be expressed in the stochastic calculus in terms of a "Wiener process" corresponding to a Gaussian random variable whose variance grows like t, as in a random walk.

Keeping terms of to order $|A|$

$$\delta N \to \left[|A|^2 + |A|\langle \hat{X}_\phi \rangle\right]\delta t + |A|\,\delta W(t) \,, \qquad \hat{X}_\phi \equiv e^{-i\phi}\hat{L} + e^{i\phi}\hat{L}^\dagger$$

Where $\delta W(t)$ is a Weiner interval, Gaussian random variable with mean 0 and variance δt: $\langle\langle \delta W(t)\rangle\rangle = 0$, $\langle\langle \delta W(t)^2\rangle\rangle = \delta t$. Thus $\delta W(t) \sim \sqrt{\delta t}$.
In the stochastic calculus, $\delta W \to dW$. One can show
$\langle\langle dW(t)\rangle\rangle = 0$ $\qquad dW^2 = dt$ $\quad \forall$ cases; no need to ensemble average.

The current seen by the Photo-detector $I(t) = \langle\langle I \rangle\rangle + \xi(t)$, where $\xi(t) = |A|\frac{dW}{dt}$
One can show $\langle\langle \xi(t)\xi(t')\rangle\rangle = |A|^2\,\delta(t-t')$: Delta-Correlated, Langevin

Our stochastic differential equation for this continuous current is:

$$d|\psi\rangle = \left[-\frac{i}{\hbar}\tilde{H}\,dt - \frac{1}{2}\left(\hat{J}^\dagger\hat{J} - \langle \hat{J}^\dagger\hat{J}\rangle\right)dt\right]|\psi(t)\rangle + \left[\frac{\hat{J}}{\|\hat{J}|\psi\rangle\|} - 1\right]dN\,|\psi(t)\rangle$$

Aside: $\dfrac{\hat{J}|\psi\rangle}{\|\hat{J}|\psi\rangle\|} = \dfrac{(\hat{L}+A)|\psi\rangle}{\|\hat{L}+A\|} = \dfrac{e^{-i\phi}\hat{L} + |A|}{\|(\hat{L}+A)|\psi\rangle\|}$ where $A = |A|e^{i\phi}$ (more convenient phase choice)

$$= \frac{e^{-i\phi}\hat{L} + A}{\sqrt{|A|^2 + |A|\langle \hat{X}_\phi\rangle + \langle \hat{L}^\dagger\hat{L}\rangle}} = \left(1 + \frac{e^{-i\phi}\hat{L}}{|A|}\right)\left(1 + \frac{\langle \hat{X}_\phi\rangle}{|A|} + \langle \hat{L}^\dagger\hat{L}\rangle\right)^{-1/2} \approx \left(1 + \frac{e^{-i\phi}\hat{L}}{|A|}\right)\left(1 - \frac{1}{2}\frac{\langle \hat{X}_\phi\rangle}{|A|} + \frac{3}{8}\frac{\langle \hat{X}_\phi\rangle^2}{|A|^2} - \frac{1}{2}\langle \hat{L}^\dagger\hat{L}\rangle\right)$$

$$\approx 1 + \frac{e^{-i\phi}\hat{L}}{|A|} - \frac{1}{2}\frac{\langle \hat{X}_\phi\rangle}{|A|} + \frac{3}{8}\frac{\langle \hat{X}_\phi\rangle^2}{|A|^2} - \frac{1}{2}\frac{e^{-i\phi}\hat{L}\langle \hat{X}_\phi\rangle}{|A|^2} - \frac{1}{2}\frac{\langle \hat{L}^\dagger\hat{L}\rangle}{|A|^2}\,, \text{ where } \hat{X}_\phi = e^{i\phi}\hat{L} + e^{+i\phi}\hat{L}$$

$$\Rightarrow \left[\frac{\hat{J}}{\|\hat{J}|\psi\rangle\|} - 1\right]dN \approx \left[\frac{e^{-i\phi}\hat{L}}{|A|} - \frac{1}{2}\frac{\langle \hat{X}_\phi\rangle}{|A|} + \frac{3}{8}\frac{\langle \hat{X}_\phi\rangle^2}{|A|^2} - \frac{1}{2}\frac{e^{-i\phi}\hat{L}\langle \hat{X}_\phi\rangle}{|A|^2} - \frac{1}{2}\frac{\langle \hat{L}^\dagger\hat{L}\rangle}{|A|^2}\right] *$$
$$\left[|A|^2 dt + |A|\langle \hat{X}_\phi\rangle dt + |A|\,dW\right]$$

Dropping terms $\mathcal{O}\left(\frac{1}{A}\right)$ and smaller

$$\left[\frac{\hat{J}}{\|\hat{J}|\psi\rangle\|} - 1\right]dN \approx A^*\hat{L}\,dt - \frac{1}{2}|A|\langle \hat{X}_\phi\rangle dt + \frac{1}{2}e^{-i\phi}\hat{L}\langle \hat{X}_\phi\rangle\,dt - \frac{1}{8}\langle \hat{X}_\phi\rangle^2 dt - \frac{1}{2}\langle \hat{L}^\dagger\hat{L}\rangle\,dt$$
$$+ \left(e^{-i\phi}\hat{L} - \frac{1}{2}\langle \hat{X}_\phi\rangle\right)dW$$

Aside: $-\frac{i}{\hbar}\tilde{H}dt - \frac{1}{2}(\hat{J}^\dagger\hat{J} - \langle\hat{J}^\dagger\hat{J}\rangle)dt =$

$$= -\frac{i}{\hbar}\hat{H}\,dt - \frac{1}{2}(A^*\hat{L} - A\hat{L}^\dagger)dt - \frac{1}{2}(\hat{L}^\dagger\hat{L} + A^*\hat{L} + A\hat{L}^\dagger - \langle\hat{L}^\dagger\hat{L}\rangle - |A|\langle\hat{X}_\phi\rangle)dt$$

$$= -\frac{i}{\hbar}\hat{H}_{eff}\,dt - A^*\hat{L}\,dt - \frac{1}{2}\hat{L}^\dagger\hat{L}dt + \frac{1}{2}\langle\hat{L}^\dagger\hat{L}\rangle dt + \frac{1}{2}|A|\langle\hat{X}_\phi\rangle dt$$

Putting this all together:

$$\boxed{d|\psi\rangle = \left(-\frac{i}{\hbar}\hat{H} - \frac{1}{2}(\hat{L}^\dagger\hat{L} - \langle\hat{X}_\phi\rangle\hat{L}e^{-i\phi} + \frac{1}{4}\langle\hat{X}_\phi\rangle^2)\right]dt\,|\psi\rangle + \left[e^{-i\phi}\hat{L} - \frac{1}{2}\langle\hat{X}_\phi\rangle\right]dW|\psi(t)\rangle}$$

This is the Stochastic Schrödinger equation for quantum trajectories associated with homodyne detection.

Check: When ensemble-averaged, we recover the Lindblad master equation

$$d\hat{\rho} = \langle\langle d(|\psi\rangle\langle\psi|)\rangle\rangle = \langle\langle |d\psi\rangle\langle\psi| + |\psi\rangle\langle d\psi| + |d\psi\rangle\langle d\psi|\rangle\rangle$$

(Note: For a Wiener process $\langle\langle dW(t)\, |\psi(t)\rangle\langle\psi(t)|\rangle\rangle = \underbrace{\langle\langle dW(t)\rangle\rangle}_{=0}\langle\langle|\psi(t)\rangle\langle\psi(t)|\rangle\rangle = 0$)

$$\Rightarrow d\hat{\rho} = -\frac{i}{\hbar}[\hat{H}_{eff},\hat{\rho}]'dt + \frac{1}{2}\langle\hat{X}_\phi\rangle(e^{-i\phi}\hat{L}\hat{\rho} + \hat{\rho}\hat{L}^\dagger e^{i\phi})dt - \frac{1}{4}\langle\hat{X}_\phi\rangle^2\hat{\rho}\,dt + \underbrace{(e^{-i\phi}\hat{L} - \tfrac{1}{2}\langle\hat{X}_\phi\rangle)\hat{\rho}(e^{+i\phi}\hat{L}^\dagger - \tfrac{1}{2}\langle\hat{X}_\phi\rangle)}_{\hat{L}\rho\hat{L}^\dagger - \frac{1}{2}\langle\hat{X}_\phi\rangle(e^{-i\phi}\hat{L}\hat{\rho} + \hat{\rho}\hat{L}e^{i\phi}) + \frac{1}{4}\langle\hat{X}_\phi\rangle^2\hat{\rho}}\overset{dt}{dW^2}$$

$$\Rightarrow \frac{d\rho}{dt} = -\frac{i}{\hbar}[\hat{H}_{eff},\hat{\rho}]' + \hat{L}\hat{\rho}\hat{L}^\dagger, \quad \text{where} \quad \hat{H}_{eff} = \hat{H} - \frac{i\hbar}{2}\hat{L}^\dagger\hat{L}$$

This is a form known as Quantum State Diffusion.

To see why this is so, consider the case where $\phi = 0$, $\hat{L} = \hat{L}^\dagger \Rightarrow \hat{X} = 2\hat{L}$

$$\Rightarrow d|\psi\rangle = \Big[-\frac{i}{\hbar}\hat{H} \underbrace{-\frac{1}{2}(\hat{L} - \langle\hat{L}\rangle)^2}_{\text{drift}}\Big]dt\,|\psi\rangle + \underbrace{(\hat{L} - \langle\hat{L}\rangle)dW\,|\psi\rangle}_{\text{diffusion}}$$

$$\frac{d|\psi\rangle}{dt} = \left(-\frac{i}{\hbar}\hat{H} - \frac{1}{2}(\hat{L} - \langle\hat{L}\rangle)^2\right)|\psi\rangle + (\hat{L} - \langle\hat{L}\rangle)|\psi\rangle\,\xi(t)$$

The SSE looks like a Langevin equation with a term that tries to damp towards an eigenstate of $\hat{L}$, in which case $(\hat{L} - \langle\hat{L}\rangle)|\psi\rangle = 0$, with Langevin noise to satisfy the "fluctuation-dissipation theorem. Note if $[\hat{H}, \hat{L}]$, the

$\hat{H}$ and $\hat{L}$ share common eigenstates. The system will dynamically evolve into an eigenstate of $\hat{L}$. This is a kind of "dynamical collapse of the wavefunction" under continuous measurement of the observable $\hat{L}$. This kind of measurement is known as a "quantum non-demolition measurement" (QND) for historical reason.

The master equation is recovered from the ensemble average of quantum trajectories.

Let $\Delta\hat{L} \equiv \hat{L} - \langle\hat{L}\rangle$: $|d\psi\rangle = \left(\frac{-i}{\hbar}\hat{H} - \frac{1}{2}\Delta\hat{L}^2\right)|\psi\rangle dt + \Delta\hat{L}|\psi\rangle dW$

$$d\hat{\rho} = \langle\langle d(|\psi\rangle\langle\psi|)\rangle\rangle = -\frac{i}{\hbar}[\hat{H},\rho]dt - \frac{1}{2}\left(\Delta\hat{L}^2\hat{\rho} + \hat{\rho}\Delta\hat{L}^2\right)dt + \Delta\hat{L}\hat{\rho}\Delta\hat{L}\underbrace{\langle\langle dW^2\rangle\rangle}_{=dt} + \left(\Delta\hat{L}\hat{\rho} + \hat{\rho}\Delta\hat{L}\right)\underbrace{\langle\langle dW\rangle\rangle}_{=0}$$

$$\Rightarrow \boxed{\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] - \frac{1}{2}[\Delta\hat{L},[\Delta\hat{L},\hat{\rho}]] = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] - \frac{1}{2}[\hat{L},[\hat{L},\hat{\rho}]]}$$

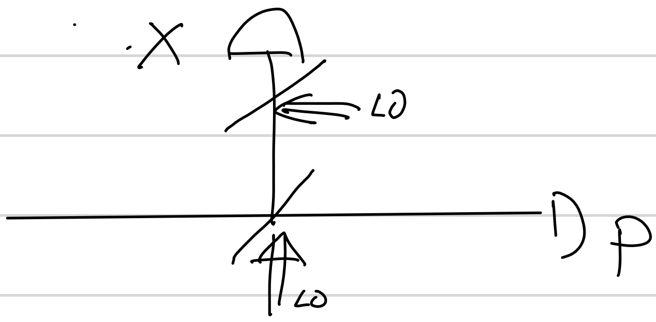
This is the Lindblad form of the master equation when $\hat{L} = \hat{L}^\dagger$, noting that

$$[\hat{L},[\hat{L},\hat{\rho}]] = \hat{L}(\hat{L}\hat{\rho} - \hat{\rho}\hat{L}) - (\hat{L}\hat{\rho} - \hat{\rho}\hat{L})\hat{L} = \hat{L}^2\hat{\rho} + \hat{\rho}\hat{L}^2 - 2\hat{L}\hat{\rho}\hat{L}$$

The idea of Quantum State Diffusion and dynamical collapse was first studied by Gisin and Percival (Phys. Rev. Lett. 52 1657 (1984), Helvetica Phys. Act 62, 363 (1989); Phys. Lett. A 143, 1 (1990); Phys. Lett. A 175 144 (1993)). The S.S.E. they wrote is

$$\boxed{|d\psi\rangle = -\frac{i}{\hbar}\hat{H}dt|\psi\rangle - \frac{1}{2}\left(\hat{L}^\dagger\hat{L} - 2\langle\hat{L}^\dagger\rangle\hat{L} + |\langle\hat{L}\rangle|^2\right)dt|\psi\rangle + (\hat{L} - \langle\hat{L}\rangle)\underbrace{dW_\alpha}_{\text{complex Wiener}}|\psi\rangle}$$

This is another form of quantum-state diffusion. This unravelling of the master equation corresponds to heterodyne detection, as opposed to homodyne detection. In heterodyne, instead of measure a quadrature $\hat{X}_\phi$, one measures both $\hat{X}$ and $\hat{P}$



This corresponds to a "complex" photon current $\langle\hat{X}\rangle + i\langle\hat{P}\rangle = \langle\hat{a}\rangle$

Note when $\hat{L} = \hat{L}^\dagger$ $d|\psi\rangle = \frac{-i}{\hbar}\hat{H}dt|\psi\rangle - \frac{1}{2}(\hat{L} - \langle\hat{L}\rangle)^2 dt|\psi\rangle + (\hat{L} - \langle\hat{L}\rangle)\left(\frac{dW_x + i\,dW_p}{\sqrt{2}}\right)|\psi\rangle$

as in the homodyne case.