Physics 581, Quantum Optics II

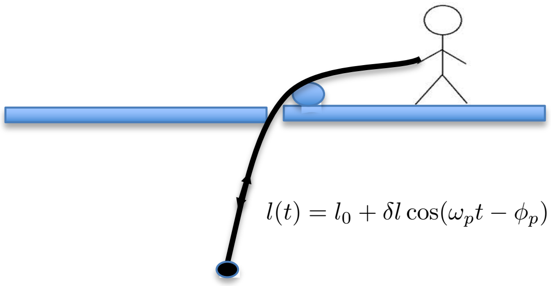
Problem Set #1

Due: Thursday, February 8, 2024

Problem 1: Parametric resonance (25 points)

Optical parametric amplification, which we have seen in the context of three-wave mixing in nonlinear optics, is a general phenomenon in nonlinear dynamics known as *parametric resonance*, which leads to squeezing. We study the basic problem here.

Consider an oscillator with a *time-dependent frequency* $\ddot{x}+\omega^2(t)x=0$. For example, consider a pendulum whose length is periodically modulated



When $\delta l << l_0$ the pendulum oscillations satisfy the Mathieu equation

$$\ddot{x}+\left(\omega_0^2+\varepsilon\cos(\omega_p t-\phi_p)\right)x=0\text{, where }\varepsilon=\frac{\delta l}{l_0}\omega_0^2\text{ and }\omega_0=\sqrt{g/l_0}$$

There is no general analytic solution to this problem. We can, however, solve this approximately. Our goal is to show that there is a nonlinear resonance, at which point we exponentially pump energy into the system.

(a) Write the general solution as $x=\mathrm{Re}\left(\alpha(t)e^{-i\omega_0 t}\right)$. For weak driving take $|\dot{\alpha}| << \omega_0|\alpha|$.

Show that

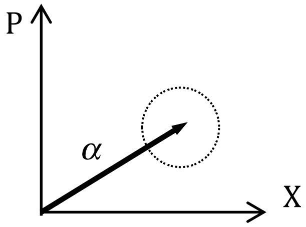
$$\dot{\alpha}-\dot{\alpha}^*e^{2i\omega_0 t}\approx -i\frac{\varepsilon}{4\omega_0}\left(\alpha e^{i\phi_P}e^{-i\omega_p t}+\alpha e^{-i\phi_P}e^{+i\omega_p t}+\alpha^*e^{-i\phi_P}e^{+i(\omega_P+2\omega_0)t}+\alpha^*e^{i\phi_P}e^{-i(\omega_P-2\omega_0)t}\right).$$

(b) This equation shows "parametric resonance" when $\omega_P=2\omega_0$ (the term "parametric" comes from the idea that we were modulating a parameter in the original oscillator). Ignoring the rapidly oscillating term, show that at parametric resonance, the equation of motion can be written in the form,

$\dot{\alpha} \approx -\kappa e^{i2\theta}\alpha^*$, where $\kappa > 0$ is real. Find the constants κ and θ. Show that the solution is $X_\theta(t) = e^{-\kappa t}X_\theta(0),\ \ P_\theta(t) = e^{+\kappa t}P_\theta(0)$, where $\alpha(t) \equiv \left(X_\theta(t) + iP_\theta(t)\right)e^{+i\theta}$.

We are all familiar with this phenomenon. As a child on a swing, we pump our legs back and forth, effectively increasing or decreasing the length of the pendulum. If we pump at twice the natural frequency we amplify our motion, but only if we pump at the right phase! In nonlinear optics, the pump laser effectively changes the optical path length of the signal and can thus parametrically amplify the signal.

(c) Parametric resonance leads to *phase-sensitive amplification*. Consider a classical statistical distribution of initial complex amplitudes.



For the conditions such that $\theta = 0$, sketch the resulting output distribution. Comment.

Problem 2: Twin beams and two-mode squeezed states. (25 points)
Considering the Hamiltonian

$$\hat{H} = i\hbar G\left(\hat{a}_+^\dagger \hat{a}_-^\dagger e^{-i\phi} - \hat{a}_+\hat{a}_- e^{i\phi}\right),$$

where $\hat{a}_\pm$ are annihilation operators for two modes with frequencies $\omega_\pm$. We will see in class how this arises in nonlinear optics through the process of parametric down-conversion. This leads to correlated twin "signal" and "idler" beams as long as the phase matching conditions are satisfied,

$$\omega_p = \omega_s + \omega_i,\ \ \mathbf{k}_p = \mathbf{k}_s + \mathbf{k}_i.$$

Here *G* is the coupling constant depending on the nonlinearity, pump amplitude, and vacuum mode strength. The state produced is known as a "two-mode squeezed vacuum state", $\hat{S}_{\pm}(\xi)|0\rangle_{+} \otimes |0\rangle_{-} = \exp\left[\xi\hat{a}_{+}\hat{a}_{-} - \xi^{*}\hat{a}_{+}^{\dagger}\hat{a}_{-}^{\dagger}\right]|0\rangle_{+} \otimes |0\rangle_{-}$, where $\xi = re^{i\phi}$ is the complex squeezing parameter for an interaction time *t*, $r = Gt$.

(a) Show that the generalized Bogoliubov transformations is

$$\hat{S}_{\pm}^{\dagger}(\xi)\hat{a}_{\pm}\hat{S}_{\pm}(\xi) = \cosh(r)\hat{a}_{\pm} - e^{-i\phi}\sinh(r)\hat{a}_{\mp}^{\dagger}.$$

(b) Show that the individual modes, $\hat{a}_{\pm}$, show no squeezing, but that squeezing exists in the *correlation* between the modes. Hint: consider quadratures,

$$\hat{X}_{\pm}(\theta) \equiv \frac{\hat{a}_{\pm}e^{i\theta} + \hat{a}_{\pm}^{\dagger}e^{-i\theta}}{2} \text{ and then } \hat{Y}(\theta,\theta') \equiv \left(\hat{X}_{+}(\theta) - \hat{X}_{-}(\theta')\right)/\sqrt{2}.$$

For the remaining parts, take ξ *real.*

(c) The two-mode squeezed state is an entangled state between the signal and idler as we know from the perturbative limit of twin photons. Show that in the Fock basis

$$\hat{S}_{\pm}(r)|0\rangle_{+} \otimes |0\rangle_{-} = \left(\cosh(r)\right)^{-1}\sum_{n=0}^{\infty}\left(\tanh(r)\right)^{n}|n\rangle_{+} \otimes |n\rangle_{-}.$$

Hint: Use the "disentangling theorem" (D. R. Traux, Phys. Rev. D **31**, 1988 (1985)):

$$e^{r\left(\hat{a}_{+}^{\dagger}\hat{a}_{-}^{\dagger} - \hat{a}_{+}\hat{a}_{-}\right)} = e^{\Gamma\hat{a}_{+}^{\dagger}\hat{a}_{-}^{\dagger}}e^{-g\left(\hat{a}_{+}^{\dagger}\hat{a}_{+} + \hat{a}_{-}^{\dagger}\hat{a}_{-} + 1\right)}e^{-\Gamma\hat{a}_{+}\hat{a}_{-}}.$$

$$\text{where } \Gamma = \tanh(r),\ g = \ln\left(\cosh(r)\right)$$

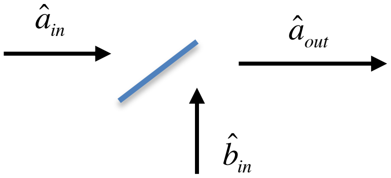
The photons are produced with perfect correlations between the modes. This is known as "number squeezing" in "twin beams.

Problem 3: Shot-noise and vacuum fluctuations (30 Points)

Squeezed states are not robust because of photon absorption (loss). There are many ways to understand this. Squeezing is associated with photon pair-correlations. Loss will randomly remove a photon, and not both in the correlated pair. The remaining photon will then just add shot noise. Another way to understand this is from the perspective of the continuous variables.

Classically, linear loss results in attenuation of the field amplitude $E \to e^{-\kappa}E = \eta E$, where η is the loss coefficient. We can model this attenuation by a partial transmitting beam splitter, with real transmission amplitude η. Quantumly, we cannot make the transformation, $\hat{a} \rightarrow \eta \hat{a}$, because the commutation relations are not preserved. Stated in another way, *we cannot attenuate the vacuum fluctuations*.

(a) Consider a linear transformation between two modes at a beam splitter with the (real) transmission amplitude η

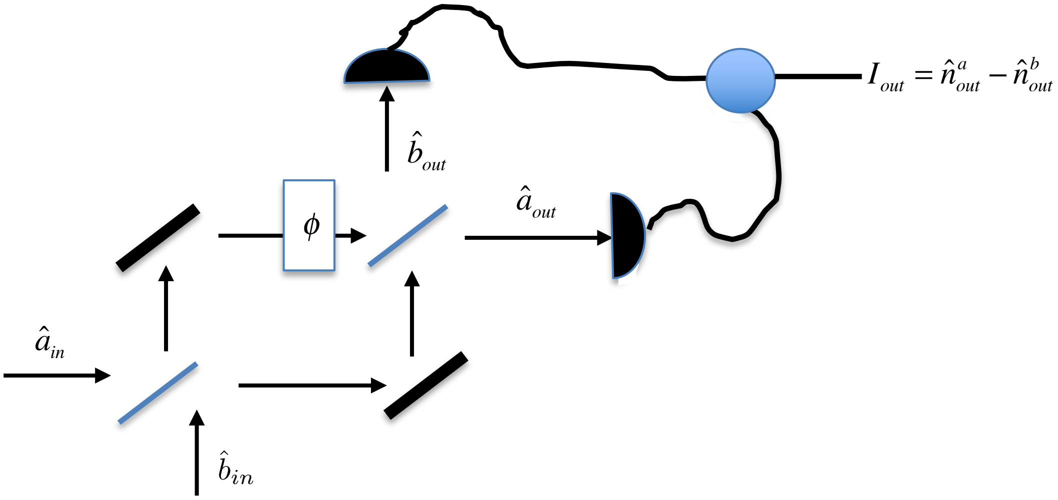


Show that if the vacuum enters in mode-b, the output fluctuations in an arbitrary quadrature of mode-a are

$$\Delta X^2_{\Theta,out} = \eta^2 \Delta X^2_{\Theta,in} + \left(1-\eta^2\right)\frac{1}{2}.$$

Interpret this result and show how squeezing is degraded by photon loss.

(b) Consider now a Mach-Zender interferometer



The 50-50 beam splitters have trans. and reflect. coefficients $t = 1/\sqrt{2}$, $r = i/\sqrt{2}$.
Show that output photocurrent operator corresponding to the difference of the photocounts at the two output ports is

$$\hat{I}_{out} = \cos\phi\left(\hat{n}_a^{in} - \hat{n}_b^{in}\right) + \sin\phi(\hat{a}^\dagger_{in}\hat{b}_{in} + \hat{b}^\dagger_{in}\hat{a}_{in}).$$

(c) If the input signal enters through port-a and vacuum in port-b, then the show the mean output signal and fluctuations are

$$\langle \hat{I}_{out} \rangle = \cos\phi \, \langle \hat{n}_a^{in} \rangle, \quad \langle \Delta \hat{I}_{out}^2 \rangle = \cos^2\phi \, \langle (\Delta \hat{n}_a^{in})^2 \rangle + \sin^2\phi \langle \hat{n}_a^{in} \rangle$$

(d) The mean output signal is what we know as the interference "fringes." The fluctuations determine the noise. The term $\cos^2\phi \, \langle (\Delta \hat{n}_a^{in})^2 \rangle$ in $\left\langle \Delta \hat{I}_{out}^2 \right\rangle$ is the contribution of the noise from the input signal. Show that the term $\sin^2\phi \langle \hat{n}_a^{in} \rangle$ arises due to the vacuum fluctuations entering from the unused port-b.

(e) In an application such as LIGO, which measures a tiny effect such as a gravity wave, we typically operate near a node of the fringe so that we are measuring an effect away from zero. Thus, we set $\phi = \pi/2 + \delta\phi$, where $\delta\phi << 1$ depends on the strength of the gravity wave. In that case, given an input coherent state in port-a, show that the signal-to-noise ratio (SNR) is

$$SNR = \frac{\langle \hat{I}_{out} \rangle}{\sqrt{\langle \Delta \hat{I}_{out}^2 \rangle}} \approx |\delta\phi| \sqrt{\langle \hat{n}_a^{in} \rangle}.$$

This is known as the "standard quantum limit" and is limited solely by the vacuum noise vacuum noise entering port-b.

(f) The seminal work on C. M. Caves, Phys. Rev. D **23**, 1693 (1981), showed that we could improve the *SNR* by injecting *squeezed vacuum* into the unused port-b, squeezed along the appropriate quadrature. Show that that in this case, the *SNR* is

$$SNR = \frac{\langle \hat{I}_{out} \rangle}{\sqrt{\langle \Delta \hat{I}_{out}^2 \rangle}} \approx e^r |\delta\phi| \sqrt{\langle \hat{n}_a^{in} \rangle},$$

where r is the squeezing parameter. Find the quadrature along which we much inject squeezed vacuum.

This is currently be implemented in the latest generation of LIGO.

http://www.nature.com/nphoton/journal/v7/n8/full/nphoton.2013.177.html