Physics 581, Quantum Optics II
Problem Set #2
Due: Thursday February 22, 2024

Problem 1: Some more boson Algebra (15 Points)

(a) Gaussian integrals in phase-space are used all the time. Show that

$$\int \frac{d^2\beta}{\pi} e^{-\gamma|\beta|^2} e^{\alpha\beta^* - \beta\alpha^*} = \frac{1}{\gamma} e^{-|\alpha|^2/\gamma} .$$

(b) Show that the displacement operators are orthogonal according to the Hilbert-Schmidt inner product, $Tr\left(\hat{D}^\dagger(\alpha)\hat{D}(\beta)\right) = \pi\delta^{(2)}(\alpha - \beta)$.

Hint: Recall $Tr\left(\hat{A}\right) = Tr\left(\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha|\hat{A}\right) = \int \frac{d^2\alpha}{\pi} \langle\alpha|\hat{A}|\alpha\rangle$

Problem 2: Ways of calculating the Wigner Function (15 points)

(a) Show that for a pure state $\hat{\rho} = |\psi\rangle\langle\psi|$, the Wigner function is

$$W(X,P) = \int_{-\infty}^{\infty} \frac{dY}{2\pi} \psi^*\left(X + \frac{Y}{2}\right)\psi\left(X - \frac{Y}{2}\right)e^{-iPY} \text{ , where } W(X,P) = \frac{1}{2}W(\alpha) .$$

This the form Wigner originally wrote in terms of the wave function.

(b) Show that

$$W(\alpha) = \frac{2}{\pi^2} e^{2|\alpha|^2} \int d^2\beta \langle -\beta|\hat{\rho}|\beta\rangle e^{-2(\beta\alpha^* - \beta^*\alpha)}$$

Problem 3: Calculation of some quasiprobability functions (30 points)

Choose 3 out of 5 parts, 10-points extra credit for all parts (a)-(e)

(a) Find and plot the *P, Q,* and *W* distributions for a thermal state

$$\hat{\rho} = \frac{e^{-\hbar\omega \hat{a}^\dagger \hat{a}/k_B T}}{Z}, Z = Tr(e^{-\hbar\omega \hat{a}^\dagger \hat{a}/k_B T}) = \text{partition function}$$

and show they are *Gaussian* functions. For example, you should find $P(\alpha) = \frac{1}{\pi\langle n\rangle}\exp\left(-\frac{|\alpha|^2}{\langle n\rangle}\right)$. Show that these three distributions give the proper functions in the limit, $\langle n\rangle \to 0$, i.e. the vacuum.

(b) Find and plot the *Q,* and *W* distributions for the squeezed state $|\psi\rangle = \hat{D}(\alpha)\hat{S}(\zeta)|0\rangle$. Does the Glauber-Sudharshan P-resprentation exist? In what sense is this state nonclassical?

(c) Find and plot the *Q,* and *W* distributions for a Fock state $|\psi\rangle = |n\rangle$. Find a formal expression for the Glauber-Sudharshan P-representation. This is not a well-behaved function. Comment.

(d) Consider a superposition state of two "macroscopically" distinguishable coherent states,

$|\psi\rangle = N(|\alpha_1\rangle + |\alpha_2\rangle)$, $|\alpha_1 - \alpha_2| >> 1$, where $N = \left[2\left(1+\exp\{-|\alpha_1 - \alpha_2|^2\}\right)\right]^{-1/2}$ is normalization.

This state is often referred to as a "Schrodinger cat", and is very nonclassical.

Find and plot the Wigner function, for the case $|\psi\rangle = N(|\alpha\rangle + |-\alpha\rangle)$, with α real, and plot it for different values of $|\alpha_1 - \alpha_2| = 2\alpha$. Comment please.

(e) Calculate the marginals of the Schrödinger-cat Wigner function in *X* and *P* and show they are what you expect.