Physics 581, Quantum Optics II

Problem Set #3
Due: Monday March 7, 2024

Choose 2 out of 3 problems. Extra credit for all three or any combination of these.

Problem 1: Gaussian States in Quantum Optics (35 points)

The set of states whose quadrature fluctuations are Gaussian distributed about a mean value is an important class in quantum optics. These states have Gaussian Wigner functions. In this problem, we explore Gaussian states, their relationship to squeezing, and the canonical algebra of phase space.

Consider a field of *n-modes*, with quadrature defined by an ordered vector:

$$\mathbf{Z} = (X_1, P_1, X_2, P_2, ..., X_n, P_n).$$

The operators associated with these quadratures satisfy a set of canonical commutators relations that can be written compactly as,

$$\left[\hat{Z}_i, \hat{Z}_j\right] = \frac{i}{2}\Sigma_{ij}, \text{ where } \Sigma = \bigoplus_{k=1}^{n}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ is a skew-symmetric matrix.}$$

We define an "inner product" in phase space as $\left(\mathbf{Z}|\mathbf{Q}\right) = Z_i \Sigma_{ij} Q_j$ (summed over repeated indices through this problem).

(a) Show that the phase space displacement operator can be written

$$\hat{D}(\mathbf{Z}) = \exp\left\{-i\left(\mathbf{Z}|\hat{\mathbf{Z}}\right)\right\}$$

A *Gaussian state* is one whose Wigner function is a Gaussian function on phase space. Recall the characteristic function of a quantum state is defined $\chi(\mathbf{Z}) = Tr\left(\hat{\rho}\hat{D}(\mathbf{Z})\right)$.
The general form of the characteristic function for a Gaussian state with is:

$$\chi(\mathbf{Z}) = \exp\left\{-\frac{1}{2}\left(\mathbf{Z}|\mathbf{C}|\mathbf{Z}\right) + i\left(\mathbf{d}|\mathbf{Z}\right)\right\}.$$

Where C_{ij} is known as the covariance matrix, and d_i is a real vector.

(b) Show that: $\left\langle \hat{Z}_i \right\rangle = d_i$, and $\frac{1}{2}\left\langle \Delta\hat{Z}_i\Delta\hat{Z}_j + \Delta\hat{Z}_j\Delta\hat{Z}_i \right\rangle = C_{ij}$, where $\Delta\hat{Z}_i \equiv \hat{Z}_i - \left\langle \hat{Z}_i \right\rangle$.

Hint: Recall how moments are found from the characteristic function.

The Gaussian state is thus determined by the mean position in phase space and the covariance of all the fluctuations.

(c) Find the Wigner function for a state with the general form of the characteristic function.

Let us restrict our attention to Gaussian states with zero mean (the mean is irrelevant to the statistics and can always be removed via a displacement operation). Consider now unitary transformations on the state. A particular class of transformations is the set that act as linear canonical transformations, i.e.

$\hat{U}^\dagger \hat{Z}_i \hat{U} = S_{ij}\hat{Z}_j$, where S_{ij} is a symplectic matrix, defined by $S^T \Sigma S = \Sigma$.

A unitary map on the state transforms the state according to

$$\chi(\mathbf{Z}) \Rightarrow \chi'(\mathbf{Z}) = Tr\left(\hat{U}\hat{\rho}\hat{U}^\dagger \hat{D}(\mathbf{Z})\right) = Tr\left(\hat{\rho}\hat{U}^\dagger \hat{D}(\mathbf{Z})\hat{U}\right).$$

(d) Show that for a symplectic transformation, the characteristic function transforms as

$$\chi(\mathbf{Z}) \Rightarrow \chi(\mathbf{SZ})$$

and thus the action of the unitary is to *preserve the Gaussian statistics, by* transforming covariance matrix as $\mathbf{C} \Rightarrow \mathbf{S}^T\mathbf{CS}$.

(e) Show that the following operations preserve Gaussian statistics:

- Linear optics: $\hat{U} = \exp\left(-i\theta_{ij}\hat{a}_i^\dagger \hat{a}_j\right)$
- Squeezing: $\hat{U} = \exp\left(\zeta_{ij}^* \hat{a}_i \hat{a}_j - \zeta_{ij}\hat{a}_i^\dagger \hat{a}_j^\dagger\right)$

(f) For each of these, show how the covariance matrix of the Gaussian transforms.

(g) Starting with the vacuum (a Gaussian state) apply the squeezing operator above. Show that the symplectic transformation on the covariant matrix leads to the expected result.

Problem 2: The EPR state (35 points)

The Einstein-Podolsky-Rosen (EPR) paradox is based around a thought experiment of measurements on an entangled state of the motion of two particles. The EPR state is a simultaneous eigenstate of relative position and the center-of-mass momentum

$$\left(\hat{X}_A - \hat{X}_B\right)|EPR\rangle = X_{rel}|EPR\rangle \ , \ \left(\hat{P}_A + \hat{P}_B\right)|EPR\rangle = P_{com}|EPR\rangle$$

The purpose of this problem is to show how one can create an approximation to this state in quantum optics, and to study their entanglement properties.

Consider now a parametric oscillator beyond the perturbative limit, where two modes (*A* and *B*) are phase matched with the pump. The resulting output state is a two-mode squeezed vacuum state

$$\left|0,0\right\rangle_r = \hat{S}_{AB}(r)\left|0\right\rangle_A\left|0\right\rangle_B = e^{r\left(\hat{a}^\dagger\hat{b}^\dagger - \hat{a}\hat{b}\right)}\left|0\right\rangle_A\left|0\right\rangle_B$$

Our goal is to show that in the limit of infinite squeezing, the is the EPR state.

(a) Show that, $\hat{S}_{AB}^\dagger\left(\hat{P}_A \pm \hat{P}_B\right)\hat{S}_{AB} = \left(\hat{P}_A \pm \hat{P}_B\right)e^{\mp r}$, and thus this operation squeezes the "relative position" and "center-of-mass momentum" quadratures.

(b) Show that $\hat{S}_{AB}^\dagger\hat{X}_A\hat{S}_{AB} = \cosh r\hat{X}_A + \sinh r\hat{X}_B$, $\hat{S}_{AB}^\dagger\hat{X}_B\hat{S}_{AB} = \cosh r\hat{X}_B + \sinh r\hat{X}_A$. This is a Heisenberg statement. (d) From this argue that, up to normalization (which is tricky)

$$\hat{S}_{AB}(r)\left|X_A\right\rangle_A\left|X_B\right\rangle_B = \left|\cosh rX_A + \sinh rX_B\right\rangle_A\left|\cosh rX_B + \sinh rX_A\right\rangle_B$$

(c) Show that the (normalized) position space wave function for the two modes is

$$\Psi_r(X_A,X_B) = \left\langle X_A\right|\left\langle X_B\right|\left|0,0\right\rangle_r = \frac{1}{\sqrt{\pi}}e^{-\frac{(X_A-X_B)^2}{4e^{-2r}}}e^{-\frac{(X_A+X_B)^2}{4e^{+2r}}} \quad \text{(plot for } r=2\text{)}$$

and in the limit of infinite squeezing $\lim_{r\to\infty}\Psi_r(X_A,X_B) \Rightarrow \delta(X_A - X_B)$

(d) By similar arguments, show that the (normalized) momentum space wave function is

$$\tilde{\Psi}_r(P_A,P_B) = \left\langle P_A\right|\left\langle P_B\right|\left|0,0\right\rangle_r = \frac{1}{\sqrt{\pi}}e^{-\frac{(P_A+P_B)^2}{4e^{-2r}}}e^{-\frac{(P_A-P_B)^2}{4e^{+2r}}} \quad \text{(plot for } r=2\text{)}$$

and in the limit of infinite squeezing $\lim_{r\to\infty}\tilde{\Psi}_r(P_A,P_B) \Rightarrow \delta(P_A + P_B)$

Thus argue that in the limit of infinite squeezing, the two-mode squeezed vacuum is the EPR state.

(e) Show that in the limit of infinite squeezing, the two-mode squeezed state can be expressed as

$$\lim_{r\to\infty}\left|0,0\right\rangle_r \Rightarrow \left|EPR\right\rangle = \int dX\left|X\right\rangle_A \otimes \left|X\right\rangle_B = \int dP\left|P\right\rangle_A \otimes \left|-P\right\rangle_B = \sum_n \left|n\right\rangle_A \otimes \left|n\right\rangle_B$$

Note: This is maximally entangled state in infinite dimensions. It is not a physical state, however, as it requires infinite energy. Nonetheless, we approximate is with large, but finite squeezing.

(f) Show that the Wigner function for the two-mode state is

$$W(X_A,P_A,X_B,P_B)=\left|\Psi_r(X_A,X_B)\right|^2\left|\tilde{\Psi}_r(P_A,P_B)\right|^2=\frac{1}{\pi^2}e^{-\frac{(X_A-X_B)^2+(P_A+P_B)^2}{2e^{-2r}}}e^{-\frac{(X_A+X_B)^2+(P_A-P_B)^2}{2e^{+2r}}}$$

(g) The Wigner function is positive, meaning there is a classical local probabilistic description of joint measurements of X_A,X_B,P_A,P_B. What are the implications for the EPR paradox and Bell's inequalities?

Problem 3: Quantum Tomography (35 points)

We have shown that Wigner function could be expressed as

$$W(\alpha)=\frac{1}{\pi}Tr\left(\hat{\rho}\hat{T}(\alpha)\right)=\frac{1}{\pi}\left\langle\hat{T}(\alpha)\right\rangle, \text{ where } \hat{T}(\alpha)=\int\frac{d^2\beta}{\pi}\hat{D}(\beta)e^{\alpha\beta^*-\beta^*\alpha}$$

(a) Show that $\hat{T}(\alpha)=\hat{D}(\alpha)\hat{T}(0)\hat{D}^\dagger(\alpha)$.

(b) Show that $\hat{T}(0)=2(-1)^{\hat{a}^\dagger\hat{a}}$. (This is not straight forward. You may assume the answer and work backwards or try to find a direct proof).

Note: the operator $(-1)^{\hat{a}^\dagger\hat{a}}=\sum_n(-1)^n|n\rangle\langle n|=\int dX|-X\rangle\langle X|$ is the "parity operator" (+1 for even parity, -1 for odd parity). Thus, we see that the Wigner function at the origin is given by the expected value of the parity.

$$W(0)=\frac{2}{\pi}Tr\left[\hat{\rho}(-1)^{\hat{a}^\dagger\hat{a}}\right]=\frac{2}{\pi}\sum_n(-1)^n\langle n|\hat{\rho}|n\rangle.$$

(c) Show that general expression

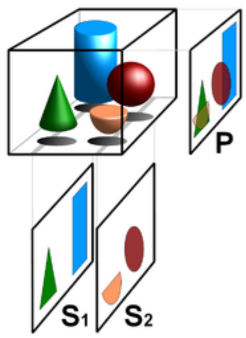
$$\hat{T}(\alpha)=2\hat{D}(\alpha)(-1)^{\hat{a}^\dagger\hat{a}}\hat{D}^\dagger(\alpha)=2\sum_n(-1)^n\hat{D}(\alpha)|n\rangle\langle n|\hat{D}^\dagger(\alpha),$$

$$\text{and thus } W(\alpha)=\frac{2}{\pi}\sum_n(-1)^n\langle n|\hat{D}^\dagger(\alpha)\hat{\rho}\hat{D}(\alpha)|n\rangle.$$

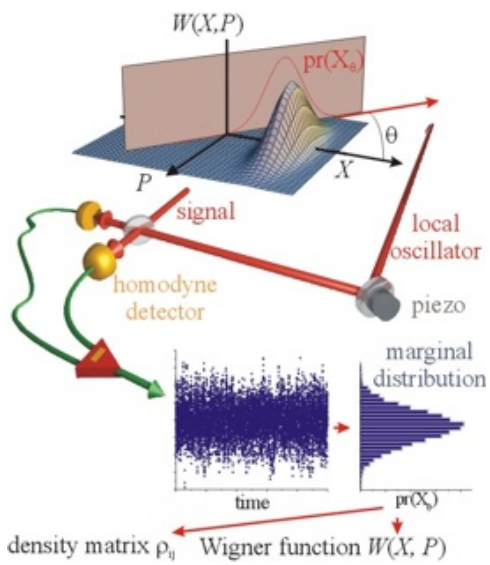
This expression provides a way to "measure" the Wigner function. One displaces the state to the point of interest, $\hat{D}^{\dagger}(\alpha)\hat{\rho}\hat{D}(\alpha)$, one then measures the photon statistics $p_{n\alpha} = \langle n|\hat{D}^{\dagger}(\alpha)\hat{\rho}\hat{D}(\alpha)|n\rangle$. Putting this in the parity sum gives $W(\alpha)$ at that point!

This is a form a quantum-state reconstruction, also known as "quantum tomography."

Form Wikipedia: "*Tomography* refers to imaging by sections or sectioning." The idea is to reconstruct a three dimensional object by a series of projections onto different planes.



The term "quantum tomography," was coined by the group of Raymer (Smithey *et al.*, PRL 70, 1244 (1993)) in the context of reconstructing the Wigner function of a light mode. This was based on the work of Vogel and Risken (PRA **40**, 2847 (1989)) who showed that the Wigner function was related to the measurements of the marginals – the projections of the Wigner function onto a plane.



We learned that a homodyne detector measures a quadrature of the field with the plane determined by the phase of the local oscillator. Thus, the measurement outcome sampled from the probability distribution, $pr(X_\theta) = \langle X_\theta | \hat{\rho} | X_\theta \rangle$, where $|X_\theta\rangle$ are the eigenstates of $\hat{X}_\theta = (\hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta})/\sqrt{2} = \hat{X}\cos\theta + \hat{P}\sin\theta$

The goal of quantum tomography is to invert and determine $\hat{\rho}$ (or equivalently $W(\alpha)$) given $\{pr(X_\theta), \forall\theta\}$.

(d) Prove the following Lemma.

Given $\delta(x)\delta(y) = \int \frac{dk_x dk_y}{(2\pi)^2} e^{-ik_x x} e^{-ik_y y}$, show that by transforming the polar coordinates in the Fourier plane,

$$\delta(X-X')\delta(P-P') = \frac{1}{(2\pi)^2}\int_0^\infty k\,dk \int_0^{2\pi} d\theta\, e^{-ik(X_\theta - X'_\theta)} = \frac{1}{(2\pi)^2}\int_{-\infty}^\infty |k|\,dk \int_0^\pi d\theta\, e^{-ik(X_\theta - X'_\theta)}.$$

(e) Starting with the trivial identity,

$$W(X,P) = \int dX'\,dP'\,\delta(X-X')\delta(P-P')W(X',P'),$$

and the result of part (a), show that the Wigner function can be obtained form the marginals by,

$$W(X,P) = \int_0^\pi d\theta \int_{-\infty}^\infty dX'_\theta\, pr(X'_\theta)K(X_\theta - X'_\theta),$$

where the integral kernel

$$K(X) = \frac{1}{4\pi^2}\int_{-\infty}^\infty dk|k|e^{-ikX} \quad .$$

This is known as the "*Radon transformation*," originally written in the context of classical tomographic image processing.

We see immediately that integral kernel is not well defined and blows up. This means that the Radon transformation is numerically unstable. In addition, it is not robust to the physical case that we have only a discrete set of measurements of $\{pr(X_\theta)\}$, and the detection is not perfect.

In the intervening decades since the original experiments in quantum tomography, the reconstruction process has been refined with more sophisticated estimation schemes based on statistical interference. This has spurred a line of research in quantum information science regarding the question of what is required to reconstruct a quantum state given finite measurement resources and noise.