Physics 581, Open Quantum Systems

Problem Set #6 EXTRA CREDIT
Due: Friday May 3, 2024

Problem 1: The transfer of coherences in the decay of an oscillator (20 points)
In the simple harmonic oscillator, the decay rate of population in a Fock state |*n*> is proportion to *n*. This makes sense physically, e.g., for a bosonic modes, as the probability of losing a photon will depend on the number of photons in the mode. However, for a coherent state, the rate of decay is independent of the amplitude! We understand this through the "transfer of coherence." Because of the equal spacing of the energy levels, *coherence fed in the master equation*. This transfer of coherences explains the way in which the mean amplitude decays. Let's examine this more closely.

(a) Consider the damped simple harmonic oscillator at finite temperature. Find the equation of motion of the off-diagonal elements (coherences) $\dot{\rho}_{n+1,n}$.

(b) Given

$$\langle \hat{a} \rangle = Tr(\hat{a}\hat{\rho}) = \sum_{n=0}^{\infty} \sqrt{n+1}\, \rho_{n+1,n}$$

show that

$$\frac{d}{dt}\langle \hat{a} \rangle = \sum_{n=0}^{\infty} \sqrt{n+1}\, \dot{\rho}_{n+1,n} = (-i\omega - \frac{\Gamma}{2})\langle \hat{a} \rangle,$$

independent of the temperature of the bath.

(c) Now consider the *nonlinear* Kerr oscillator, with Hamiltonian

$$\hat{H} = \hbar\omega \hat{a}^\dagger \hat{a} + \frac{1}{2}\hbar\kappa \hat{a}^{\dagger 2} \hat{a}^2$$

Including the same Lindbladian for the damped SHO at zero temperature, find the equation of motion of $\dot{\rho}_{n+1,n}$, and from this show by the same procedure in (b)

$$\frac{d}{dt}\langle \hat{a} \rangle = (-i\omega - \frac{\Gamma}{2})\langle \hat{a} \rangle - i\kappa \langle \hat{n}\hat{a} \rangle$$

(d) Repeat this calculation by directly finding $d\langle \hat{a} \rangle / dt$ from the commutators.
Thus, for the nonlinear oscillator the transfer of coherences is not perfect as the spacing between levels is not equal. Thus, the decay of the oscillator depends on its amplitude.

(e) But wait a minute! How did we get away with the same Lindbladian? Why are any transfer of coherences included in the first place, since the spacing between levels is not equal. Why didn't we need to include the Kerr Hamiltonian in the derivation of the Lindbladian?

Problem 2: Decoherence and cat states (35 points)

Consider the damped simple harmonic oscillator governed by the master equation

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] - \frac{\Gamma}{2}(\hat{a}^\dagger \hat{a}\rho + \rho \hat{a}^\dagger \hat{a}) + \Gamma \hat{a}\rho\hat{a}^\dagger$$, where $\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a}$.

(a) Suppose we prepare an initial pure coherent state, $\hat{\rho}(0) = |\alpha\rangle\langle\alpha|$. From the master equation,

$$\hat{\rho}(dt) = \left(\hat{1} - \frac{i}{\hbar}\hat{H}_{\text{eff}}\,dt\right)\hat{\rho}(0)\left(\hat{1} + \frac{i}{\hbar}\hat{H}^\dagger_{\text{eff}}\,dt\right) + \Gamma dt\ \hat{a}\hat{\rho}(0)\hat{a}^\dagger \quad \hat{H}_{\text{eff}} = \hat{H} - i\frac{\hbar\Gamma}{2}\hat{a}^\dagger\hat{a}$$

Show that $\hat{\rho}(dt) = |\alpha e^{-i\omega dt}e^{-\Gamma dt/2}\rangle\langle\alpha e^{-i\omega dt}e^{-\Gamma dt/2}|$ (Hint keep terms only to order *dt*). From this argue that for finite time $\hat{\rho}(t) = |\alpha(t)\rangle\langle\alpha(t)|$, where $\alpha(t) = \alpha e^{-i\omega t}e^{-\Gamma t/2}$, i.e., the state remains a pure coherent state for all times, with a complex amplitude that follows the classical trajectory.

(b) Now consider and initial pure state which is a superposition of coherent states $|\psi(0)\rangle = \mathcal{N}\left(|\alpha\rangle + |\beta\rangle\right)$, where $\mathcal{N}^{-2} = 2(1 + \text{Re}\langle\alpha|\beta\rangle)$ is the normalization. Consider the different map generated by the master equation. Show that,

$$\begin{aligned}\mathcal{E}(dt)\left[|\alpha\rangle\langle\beta|\right] &= \left(\hat{1} - \frac{i}{\hbar}\hat{H}_{\text{eff}}\,dt\right)|\alpha\rangle\langle\beta|\left(\hat{1} + \frac{i}{\hbar}\hat{H}^\dagger_{\text{eff}}\,dt\right) + \Gamma dt\ \hat{a}|\alpha\rangle\langle\beta|\hat{a}^\dagger \\ &= \langle\beta|\alpha\rangle^{\Gamma dt}\ |\alpha e^{-i\omega dt}e^{-\Gamma dt/2}\rangle\langle\beta e^{-i\omega dt}e^{-\Gamma dt/2}|\end{aligned}$$

(c) From this show that for an initial superposition of coherent states, the solution to the master equation is

$$\hat{\rho}(t) = \mathcal{N}^2\left(|\alpha(t)\rangle\langle\alpha(t)| + f(t)|\alpha(t)\rangle\langle\beta(t)| + f^*(t)|\beta(t)\rangle\langle\alpha(t)| + |\beta(t)\rangle\langle\beta(t)|\right)$$

$$\alpha(t) = \alpha e^{-i\omega t}e^{-\Gamma t/2},\ \beta(t) = \beta e^{-i\omega t}e^{-\Gamma t/2},\ f(t) = \langle\beta|\alpha\rangle^{(1-e^{-\Gamma t})}.$$

(d) Given the result (c) write the Wigner function for the state as a function time.

(e) For a Schrödinger cat state with $\beta = -\alpha$ and $|\alpha| >> 1$, qualitatively describe the time evolution of the state.

(f) (Extra credit 10 points). Make a move of the time evolution of the cat state evolving according to master equation for three cases: α=1,4,10. For simplicity, go into the rotating frame and by setting ω=0. Comment on your results.

Problem 3: Nonclassical light generation via the Kerr effect. (35 points)

In the classical (optical) Kerr effect, the index of refraction is proportional to the intensity. The quantum optical description is via the Hamiltonian,

$$\hat{H} = \frac{\hbar\kappa}{2} : \hat{I}^2 := \frac{\hbar\kappa}{2}\hat{a}^{\dagger 2}\hat{a}^2.$$

(a) Suppose the initial state is a coherent state $|\alpha\rangle$. *Linearize* this Hamiltonian about the mean field via the substitution $\hat{a} = \alpha + \hat{b}$, and keep terms only up to quadratic order in $\hat{b}$ and $\hat{b}^\dagger$. Show that the resulting Hamiltonian leads to squeezing. What quadrature is squeezed?

(b) Now let's go beyond the linear approximation. Show that for a long time such that $\kappa t = \pi$, the state becomes a Schrödinger cat, $\left(e^{i\pi/4}|-i\alpha\rangle + e^{-i\pi/4}|i\alpha\rangle\right)/\sqrt{2}$ (Hint: consider i^n for even and odd n and find the periodic pattern).

(c) Show that the Wigner function evolves in time according to the following partial differential equation

$$\frac{\partial W}{\partial t} = -i\kappa(|\alpha|^2 - 1)\left(\alpha^*\frac{\partial W}{\partial \alpha^*} - \alpha\frac{\partial W}{\partial \alpha}\right) - \frac{i\kappa}{4}\left(\alpha^*\frac{\partial}{\partial \alpha} - \alpha\frac{\partial}{\partial \alpha^*}\right)\frac{\partial^2 W}{\partial \alpha \partial \alpha^*}.$$

What is the equation in the Truncated Wigner Approximation (TWA)? Comment on the physical interpretation.

(d) A tough problem, but worth a try: Using a decomposition of the stat in the Fock basis, write an analytic expression for Wigner function as a function of time. Now take $\alpha=4$. Numerically make a movie of $W(\alpha,t)$ for $0 \le t \le 2\pi/\kappa$. (you will have to appropriately truncate the Fock space). Please comment on the results.