Physics 581: Open Quantum Systems
Lecture 15: Heisenberg Picture

We have primarily studied open quantum systems in the Schrödinger picture, by looking at the time evolution of the quantum state $\hat{\rho}(t)$. What about the Heisenberg picture? Let us begin by reviewing these pictures in closed quantum systems. In that case everything is governed by unitary evolution according to the time-dependent Schrödinger equation

$$\frac{\partial \hat{U}(t)}{\partial t} = \frac{-i}{\hbar} \hat{H}(t) \hat{U}(t)$$

where here we have allowed for a general time-dependent Hamiltonian, so the solution is

$$\hat{U}(t) = \mathcal{T}\left[\exp \frac{-i}{\hbar} \int_0^t \hat{H}(t')dt'\right]$$

In quantum mechanics, all we do is calculate probabilities for measurement outcomes

$$P_\mu(t) = \mathrm{Tr}(\hat{\rho}\, \hat{E}_\mu)(t) \quad \text{where } \{\hat{E}_\mu\} \text{ is a POVM}$$

In the <u>Schrödinger picture</u> $P_\mu(t) = \mathrm{Tr}(\hat{\rho}(t)\, \hat{E}_\mu)$ where $\hat{\rho}(t) = \hat{U}(t)\hat{\rho}\,\hat{U}^\dagger(t)$

In the <u>Heisenberg picture</u> $P_\mu(t) = \mathrm{Tr}(\hat{\rho}\, \hat{E}_\mu(t))$ where $\hat{E}_\mu(t) = \hat{U}^\dagger(t)\, \hat{E}_\mu\, \hat{U}(t)$

Typically we consider measurement in a basis corresponding to the eigenvectors of a Hermitian observable $\hat{A}$. And we consider expectation values

$$\langle \hat{A} \rangle = \mathrm{Tr}(\hat{\rho}\hat{A}) = \sum_a a\, \mathrm{Tr}(\hat{\rho}\, \hat{E}_a) \quad \text{where} \quad \hat{E}_a = |a\rangle\langle a|$$

Then $\langle \hat{A} \rangle = \mathrm{Tr}(\hat{\rho}(t)\, \hat{A}) = \mathrm{Tr}(\hat{\rho}\, \hat{A}(t))$ where $\hat{A}(t) = \hat{U}^\dagger(t)\, \hat{A}\, \hat{U}(t)$ in Heisenberg pict.

The Heisenberg picture is useful when we are interested in specific "observables" $\hat{A}$, and typically easier to solve for complex systems compared to the Schrödinger picture, which allows us to calculate an arbitrary expectation values. The Heisenberg picture has a more direct connection to classical dynamics and the Hamilton's equations of motion.

In the Schrödinger picture, the state evolve in time according to the von Neumann equation

$$\frac{\partial \hat{\rho}}{\partial t} = \frac{\partial \hat{U}}{\partial t}\hat{\rho}\,\hat{U}^{+} + \hat{U}\hat{\rho}\frac{\partial \hat{U}^{+}}{\partial t} = \frac{-i}{\hbar}[\hat{H},\hat{\rho}]$$

The Heisenberg equation of motion for observables

$$\frac{\partial \hat{A}}{\partial t} = \frac{\partial \hat{U}^{+}}{\partial t}\hat{A}\,\hat{U} + \hat{U}^{+}\hat{A}\frac{\partial \hat{U}}{\partial t} = \frac{-i}{\hbar}[\hat{A},\hat{H}]$$

Note: $(\hat{A}\hat{B})(t) = \hat{U}^{+}\hat{A}\hat{B}\hat{U} = \hat{U}^{+}\hat{A}\hat{U}\,\hat{U}^{+}\hat{B}\hat{U} = \hat{A}(t)\,\hat{B}(t)$

In general $f(\hat{A},t) = f(\hat{A}(t),0)$ in the closed-system Heisenberg picture.

Open System

Are similarily interested in Markovian CP-maps $\mathcal{E}(t)$ governed by the Lindblad equation $\frac{\partial \mathcal{E}(t)}{\partial t} = \mathcal{L}(t)\,\mathcal{E}(t)$, $\mathcal{L}(t)$ = Linbladian (generally time dependent)

$$\mathcal{E}(t) = \mathcal{T}\left[\exp \int_0^t \mathcal{L}(t')\,dt'\right] \quad \text{(super operator)}$$

Expectation value $\langle \hat{A}\rangle(t) = (\hat{A}|\,\mathcal{E}(t)\,|\hat{\rho})$

Schrödinger picture: $|\hat{\rho}(t)) = \mathcal{E}(t)|\hat{\rho}(0))$

$$\frac{\partial}{\partial t}|\hat{\rho}(t)) = \frac{\partial \mathcal{E}(t)}{\partial t}|\hat{\rho}(0)) = \mathcal{L}(t)\,\mathcal{E}(t)|\hat{\rho}(0)) = \mathcal{L}(t)\,|\hat{\rho}(t))$$

Linblad operators

or $$\frac{\partial}{\partial t}\hat{\rho}(t) = \frac{-i}{\hbar}[\hat{H}(t),\hat{\rho}(t)] - \frac{1}{2}\sum_{\mu}\{\hat{L}^{+}_{\mu}(t)\hat{L}_{\mu}(t),\hat{\rho}(t)\} + \sum_{\mu}\hat{L}_{\mu}(t)\,\hat{\rho}(t)\,\hat{L}^{+}_{\mu}(t)$$

(Linblad Master equation)

Heisenberg picture: $(\hat{A}(t)| = (\hat{A}(0)|\,\mathcal{E}(t) \Rightarrow |\hat{A}(t)) = \mathcal{E}^{+}(t)\,|\hat{A}(0))$ Adjoint representation

$$\frac{\partial}{\partial t}|\hat{A}(t)) = \frac{\partial \mathcal{E}^{+}(t)}{\partial t}|\hat{A}(0)) = \mathcal{E}^{+}(t)\,\mathcal{L}^{+}(t)\,|\hat{A}(0)).$$

This is generally not a differential equation for $|\hat{A}(t))$ making this equation generally not useful.

The exception is when $[\mathcal{E}(t), \mathcal{L}(t)]$ (e.g. $\mathcal{L}$ is independent of time)

Then $\frac{d}{dt} |\hat{A}(t)) = \mathcal{L}^{\dagger}(t) \, \mathcal{E}^{\dagger}(t) |\hat{A}(0)) = \mathcal{L}^{\dagger}(t) \, |\hat{A}(t))$

$$\text{or} \quad \frac{d}{dt}\hat{A}(t) = -\frac{i}{\hbar}[\hat{A}(t), \hat{H}] - \frac{1}{2}\sum_{\mu}\{\hat{L}^{\dagger}_{\mu}\hat{L}_{\mu}, \hat{A}(t)\} + \sum_{\mu}\hat{L}^{\dagger}_{\mu}\hat{A}(t)\hat{L}_{\mu}$$

Note: Unlike for a closed system, $(\hat{A}\hat{B})(t) \neq \hat{A}(t)\,\hat{B}(t)$
and $f(\hat{A}, t) \neq f(\hat{A}(t), 0)$

For example: In the adjoint representation $\frac{d\hat{a}}{dt} = \left(-i\omega - \frac{\Gamma}{2}\right)\hat{a} \Rightarrow \hat{a}(t) = \hat{a}(0)\, e^{-i\omega t - \frac{\Gamma t}{2}}$

$$\Rightarrow [\hat{a}(t), \hat{a}^{\dagger}(t)] = [\hat{a}(0), \hat{a}^{\dagger}(0)]\, e^{-\Gamma t} = e^{-\Gamma t}$$

Or equivalently $[\hat{X}(t), \hat{P}(t)] = i\, e^{-\Gamma t}$

The canonical commutation relations are not preserved in time. Thus, while the adjoint representation is use for calculating the expectation value of a given observable, we must be careful to understand its interpretation, and it is often cumbersome when dealing with complex systems

Heisenberg-Langevin Approach

The difficulty in the Heisenberg equations of motion in the adjoint representation stem from the fact that we have traced over the enviroment. An approach which preserves unitarity, but also the Markov approximation is known as the Heisenberg-Langevin equations. To understand this, we must first revisit the classical statistical mechanics of an open system.

Liouville vs. Langevin

We have seen that in classical statistical mechanics, a probability distribution on phase space evolves according to the Liouville equation

$$\frac{\partial P}{\partial t}(\vec{q}, \vec{p}, t) = \{H, P\}_{P.B.} + \mathcal{L}[P]$$

where the first term is the Hamiltonian evolution and the second arises from the coupling to the environment for which we have incomplete information.

For a closed classical system

$$\frac{\partial P}{\partial t} = \{H, P\} \quad \text{where} \quad \overline{A(\vec{q},\vec{p})}(t) = \int d\vec{q}\, d\vec{p}\; P(\vec{q},\vec{p},t)\, A(\vec{q},\vec{p}) \quad \text{(Liouville picture)}$$

Alternatively we can look at trajectories in the "Hamilton picture"

$$\frac{\partial q}{\partial t} = \{q, H\} = \frac{\partial H}{\partial p}, \quad \frac{\partial p}{\partial t} = \{p, H\} = -\frac{\partial H}{\partial q}$$

Or more generally, $\frac{\partial A}{\partial t} = \{A, H\} \Rightarrow \overline{A(q,p)}(t) = \int d\vec{q}\, d\vec{p}\; P(\vec{q},\vec{p})\, A(\vec{q},\vec{p},t)$

$$= \int d\vec{q}\, d\vec{p}\; P(\vec{q},\vec{p})\, A(\vec{q}(t), \vec{p}(t), 0)$$

Solving for $\vec{q}(t)$, $\vec{p}(t)$ solves the problem.

$$P(\vec{q},\vec{p},t) = P(\vec{q}(-t), \vec{p}(-t), 0)$$

For the open system we do not have full information to predict the trajectories $(\vec{q}(t), \vec{p}(t))$. The trajectories are <u>stochastistic</u>. Typically

$$\frac{d\vec{q}}{dt} = \vec{p} \qquad \frac{d\vec{p}}{dt} = \vec{F}(t) = \langle \vec{F}(t) \rangle + \vec{\mathcal{F}}(t)$$

Here $\langle \vec{F}(t) \rangle$ is the mean force, averaged over the appropriate statistical distribution, and $\vec{\mathcal{F}}(t) = \vec{F} - \langle \vec{F}(t) \rangle$ is the fluctuating "Langevin force" with $\langle \vec{\mathcal{F}}(t) \rangle = 0$

In the Langevin picture, we calculate statistical averages

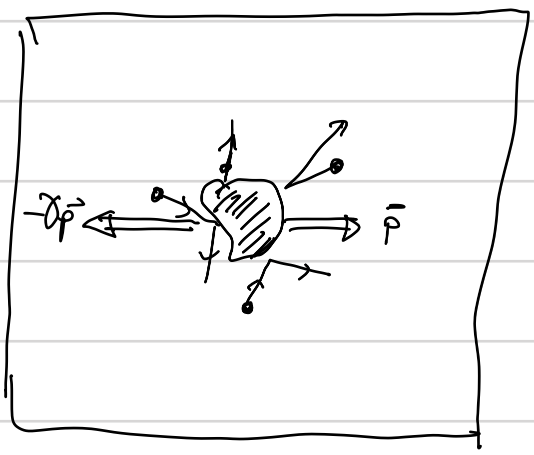
$$\overline{A(q,p)}(t) = \left\langle \int dq\, dp\; P(q,p)\, A(q(t), p(t)) \right\rangle$$ where the average $\langle\ \rangle$ is

over the statistical properties of the fluctuating force.

Alternatively, in the Liouville picture, due to the unknown force, the probability distribution no longer evolve according to a Hamiltonian solely, but now has a term relating to the coupling to the environment

Brownian Motion

Of particular interest is the Langenvin equation for Brownian motion, which describes the random irregular motion of a particle suspended in a liquid, originally observed by Robert Brown in 1827. Einstein, in 1905, explained the phenomenon through a molecular description of the fluid. It was one of the most important results that helped establish the atomist picture of matter in the 19th→20th century.



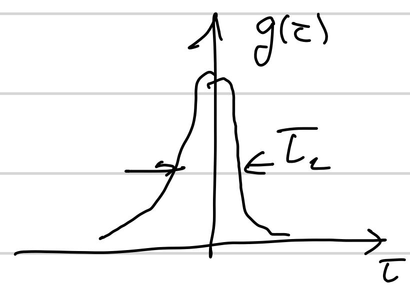
The random impact of molecules in the fluid on a pollen grain causes both drag (viscosity) on average, and diffusion due to fluctuations

$\langle F(t)\rangle = -\gamma p$ (drag) $\mathcal{F}(t)$ = Fluctuating Langenvin force

$\Rightarrow$ Langenvin Equation $\quad \frac{dp}{dt} = -\frac{\Gamma}{2}p + \mathcal{F}(t)$ (Assuming here no additional external force)

The Langenvin force has certain statistical properties

$\langle \mathcal{F}(t)\rangle = 0$ (by definition) $\langle \mathcal{F}(t_1)\mathcal{F}(t_2)\rangle = g(t_1 - t_2)$ (Stationarity of statistics)



The characteristic time scale τ_c is the fluctuating force correlation time. After that time the force is completely random.

τ_c represents the time in which the motion of the molecules is correlated with the motion of the pollen grain. After that time, the molecules rethermalize with the bath, and all "memory" of their motion related to the system is lost. This time will be very short compared to the time over which the collective effect of the molecules affects the motion of pollen grain, so

$$\tau_c \ll \frac{1}{\Gamma}$$

In the Markov approximation, valid when $\tau_c \lll \frac{1}{\Gamma}$ $\tau_c \approx 0$

And thus $\langle \mathcal{F}(t)\mathcal{F}(t')\rangle = D\,\delta(t-t')$ where D is the proprotionality

The Langenvin force is "delta correlated" corresponding to white noise

Under these assumptions we can solve for statisitcal averages of moments.
First, formally integrate the Langenvin equation

$$p(t) = p(0)e^{-\Gamma t/2} + \int_0^t dt'\, e^{-\frac{\Gamma}{2}(t-t')}\mathcal{F}(t')$$

$$\Rightarrow \langle p(t)\rangle = p(0)\,e^{-\Gamma t/2}$$

$$\langle \Delta p^2(t)\rangle = \langle (p(t)-\langle p(t)\rangle)^2\rangle = \int_0^t dt' \int_0^t dt'' \; e^{\frac{\Gamma}{2}(t'+t'')}\underbrace{\langle \mathcal{F}(t')\mathcal{F}(t'')\rangle}_{D\,\delta(t'-t'')}\; e^{-\Gamma t}$$

$$\langle \Delta p^2(t)\rangle = D e^{-\Gamma t}\int_0^t dt'\, e^{\Gamma t'} = \frac{D}{\Gamma}\left(1-e^{-\Gamma t}\right)$$

For short times, $\Gamma t \ll 1$ $\langle \Delta p^2(t)\rangle = Dt$ $\Rightarrow$ D = Diffusion coefficient

As $\Gamma t \to \infty$ $\langle \Delta p^2\rangle_{s.s} = \frac{D}{\Gamma}$ (Steady state)

Assume the fluid is a thermal bath, then the pollen comes to equilibrium
In thermal equilibrium $\langle p\rangle_{s.s.} = 0$ $\left\langle \frac{p^2}{2m}\right\rangle_{s.s.} = \frac{\langle \Delta p^2\rangle_{s.s}}{2m} = \frac{1}{2}k_B T$

$$\Rightarrow \langle \Delta p^2\rangle_{s.s.} = mk_B T = \frac{D}{\Gamma} \Rightarrow \boxed{D = \Gamma m k_B T}$$

The relationship between the diffusion coefficient and the damping rate is known in classical statistical physics as the fluctuation-dissipation theorem. Physically, this makes sense as both diffussion and damping arising from the same source — the random coupling to the bath. One can't have one without the other. Classically, this goes to zero as $T \to 0$. Quantumly, we will see that there is still a source of damping due to quantum fluctuations.

Heisenberg - Langevin Equation

Our goal is the formulate a quantum description of the Langenvin equations of motion. This is the Heisenberg picture of a coupling to a bath in the Markov approximation. We start with a full description of the system & bath with Hamiltonian $\hat{H} = \hat{H}_S + \hat{H}_B + \hat{H}_{int}$

In the interaction Schrödinger picture, given an interaction of the form

$$\hat{H}_{int}(t) = \hbar\left(\hat{L}^{\dagger}\hat{F}(t) + \hat{L}\hat{F}^{\dagger}(t)\right)$$

In the Markov approximation $\hat{\rho}_{SB}(t) = \hat{\rho}_S(t) \otimes \hat{\rho}_B(0)$ in coarse grained time $\Delta t \gg \tau_c$

$$\frac{\Delta\rho_S}{\Delta t} \approx -\int_0^{\infty} d\tau \Big\{\left(\hat{L}^{\dagger}\hat{L}\,\hat{\rho}_S(t+\tau) + \hat{\rho}_S(t+\tau)\hat{L}^{\dagger}L - 2\hat{L}\,\hat{\rho}_S(t+\tau)\hat{L}^{\dagger}\right)\langle\hat{F}(\tau)\hat{F}^{\dagger}(0)\rangle$$
$$+\left(\hat{L}\hat{L}^{\dagger}\hat{\rho}_S(t+\tau) + \hat{\rho}_S(t+\tau)\hat{L}\hat{L}^{\dagger} - 2\hat{L}^{\dagger}\hat{\rho}_S(t+\tau)\hat{L}\right)\langle\hat{F}^{\dagger}(\tau)\hat{F}(0)\rangle\Big\}$$

where $\hat{F}(t)$ is the reservoir <u>noise operator</u>. In the Markov approximation for standard models where the reservoir is a thermal bath of oscillators

$$\langle\hat{F}^{\dagger}(t_1)\hat{F}(t_2)\rangle = \frac{\Gamma}{2}\bar{n}\,\delta(t_1-t_2), \qquad \langle\hat{F}(t_1)\hat{F}^{\dagger}(t_2)\rangle = \frac{\Gamma}{2}(\bar{n}+1)\,\delta(t_1-t_2)$$

where $\bar{n} = \dfrac{1}{e^{+\hbar\omega/kT} - 1} \approx \dfrac{kT}{\hbar\omega}$ at high temperature

In the Heisenberg picture we seek the equation of motion for the system observables interacting with the reservoir. Unlike in the adjoint representation, we will not trace out the bath, so over time, these observable act on the joint system + bath Hilbert space. However, we do not seek the exact equation of motion for all coupled degrees of freedom. Our goal is to derive the equations of motion for the system observables, where the bath observables are treated statistically, leading to a fluctuating Langenvin force, under the Markov approximation.

As an example, consider the damped SHO modeled as the single oscillator coupled to a thermal bath of other oscillators

$$\hat{H} = \hat{H}_S + \hat{H}_B + \hat{H}_{int}$$

where

$$\hat{H}_S = \hbar\omega_0 \hat{a}^\dagger \hat{a}, \quad \hat{H}_B = \sum_j \hbar\omega_j \hat{b}_j^\dagger \hat{b}_j, \qquad \hat{H}_{int} = \underbrace{\sum_j{}' \hbar(\kappa_j \hat{a}^\dagger \hat{b}_j + \kappa_j^* \hat{a}\hat{b}_j^\dagger)}_{RWA}$$

The Heisenberg equations of motion for the oscillators are

$$\frac{d\hat{a}}{dt} = -i\omega_0 \hat{a} - i\sum_j{}' \kappa_j \hat{b}_j \quad , \qquad \frac{d}{dt}\hat{b}_j = -i\omega_j \hat{b}_j - i\kappa_j^* \hat{a}$$

We can formally integrate the bath oscillators

$$\hat{b}_j(t) = \hat{b}_j(0)e^{-i\omega_j t} - i\kappa_j^* \int_0^t dt' e^{-i\omega_j(t-t')} \hat{a}(t')$$

Substituting back, we obtain an integrodifferential equation for $\hat{a}(t)$

$$\frac{d\hat{a}}{dt} = -i\omega_0 \hat{a} - i\sum_j \kappa_j \hat{b}_j e^{-i\omega_j t} - \int_0^t dt' \sum_j{}' |\kappa_j|^2 e^{-i\omega_j(t-t')} \hat{a}(t')$$

Let's go to the rotating frame. Let $\hat{a}(t) = \tilde{a}(t)e^{-i\omega_0 t}$, $\quad \tilde{a}(t) = \hat{a}(t)e^{+i\omega_0 t}$

$$\frac{d}{dt}\tilde{a} = -\int_0^t dt' \sum_j{}' |\kappa_j|^2 e^{-i\Delta_j(t-t')} \tilde{a}(t') + \hat{\tilde{f}}(t)$$

where $\boxed{\hat{\tilde{f}}(t) = -i\sum_j{}' \kappa_j \hat{b}_j(0) e^{-i\Delta_j t} \equiv \text{Langevin noise operator}}$

With $\sum_j \Rightarrow \int d\omega_j \, \mathcal{D}(\omega_j)$, and under the usual Born approximation

density of states

$$\sum_j |\kappa_j|^2 e^{-i\Delta_j t} \approx \frac{\Gamma}{2}\int_0^\infty d\Delta_j \, e^{-i\Delta_j(t-t')} = \frac{\Gamma}{2}\delta(t-t') + i\delta\omega$$

shift

$$\Rightarrow \boxed{\begin{array}{l} \dfrac{d\tilde{a}}{dt} = -\dfrac{\Gamma}{2}\tilde{a}(t) + \hat{\tilde{f}}(t) \quad \text{(Heisenberg-Langevin equations)} \\[2ex] \langle \hat{\tilde{f}}^\dagger(t)\hat{\tilde{f}}(t')\rangle_R = \dfrac{\Gamma}{2}\bar{n}\,\delta(t-t') \qquad \langle \hat{\tilde{f}}(t)\hat{\tilde{f}}^\dagger(t')\rangle_R = \dfrac{\Gamma}{2}(\bar{n}+1)\,\delta(t-t') \end{array}}$$

In contrast to the adjoint representation, the inclusion of the noise operators, representing the coupling to bath satisfies the fluctuation-dissipation theorem, thereby preserving the commutation relations

Formally integrating the Langenvin equation

$$\tilde{a}(t) = \tilde{a}(0)e^{-\frac{\Gamma t}{2}} + \int_0^t dt'\, e^{-\frac{\Gamma}{2}(t-t')}\,\hat{\mathcal{F}}(t')$$

$$\Rightarrow [\tilde{a}(t), \tilde{a}^\dagger(t)] = \underbrace{[\tilde{a}(0),\tilde{a}^\dagger(0)]e^{-\Gamma t}}_{\text{Damped fluctuations}} + \underbrace{\int_0^t dt'\, dt''\, e^{\frac{\Gamma}{2}(t'+t'')}[\tilde{\mathcal{F}}(t'),\tilde{\mathcal{F}}^\dagger(t'')]\, e^{-\Gamma t}}_{\text{Source fluctuations from bath}}$$

$$\Rightarrow \langle[\tilde{a}(t), \tilde{a}^\dagger(t)]\rangle_R = e^{-\Gamma t}\Big(\underbrace{[\tilde{a}(0), \tilde{a}^\dagger(0)]}_{1} + \int_0^t dt'\, dt''\, e^{\frac{\Gamma}{2}(t'+t'')}\underbrace{\langle[\hat{\mathcal{F}}(t'), \hat{\mathcal{F}}^\dagger(t'')]\rangle}_{=\Gamma\delta(t'-t'')}\Big)$$

$$\boxed{\Rightarrow \langle[\tilde{a}(t), \tilde{a}^\dagger(t)]\rangle_R = e^{-\Gamma t}\Big(1 + \Gamma\int_0^t dt' e^{\Gamma t'}\Big) = 1 \qquad \text{Commutator preserved!}}$$

Generalized Einstein relations

$$\frac{d}{dt}\langle\tilde{a}\rangle_R = -\frac{\Gamma}{2}\langle\tilde{a}\rangle_R + \cancel{\langle\hat{\mathcal{F}}\rangle_R}^{\,0} \quad \Rightarrow \quad \boxed{\langle\hat{a}\rangle_R(t) = e^{-\frac{\Gamma}{2}t}\langle\hat{a}\rangle_R(0)}$$

$$\frac{d}{dt}\langle\hat{n}\rangle_R = \frac{d}{dt}\langle\tilde{a}^\dagger\tilde{a}\rangle_R = \Big\langle\frac{d\tilde{a}^\dagger}{dt}\tilde{a}\Big\rangle_R + \Big\langle\tilde{a}^\dagger\frac{d\tilde{a}}{dt}\Big\rangle_R = -\Gamma\langle\hat{n}\rangle_R + \langle\hat{\mathcal{F}}^\dagger(t)\,\tilde{a}(t) + \tilde{a}^\dagger(t)\,\hat{\mathcal{F}}(t)\rangle$$

$$= -\Gamma\langle\hat{n}\rangle_R + \underbrace{\cancel{\langle\hat{\mathcal{F}}^\dagger(t)\rangle}}_{0}\,\tilde{a}(0)e^{-\Gamma t} + \underbrace{\cancel{\langle\hat{\mathcal{F}}(t)\rangle}}_{0}\,\tilde{a}^\dagger(0)\,e^{-\frac{\Gamma}{2}t} + \int_0^t dt'\,\langle\hat{\mathcal{F}}^\dagger(t)\,\hat{\mathcal{F}}(t') + \hat{\mathcal{F}}^\dagger(t')\,\hat{\mathcal{F}}(t)\rangle\, e^{-\frac{\Gamma}{2}(t-t')}$$

Aside: $\int_0^t dt'\,\langle\hat{\mathcal{F}}^\dagger(t)\,\hat{\mathcal{F}}(t') + \hat{\mathcal{F}}^\dagger(t')\,\hat{\mathcal{F}}(t)\rangle\, e^{-\frac{\Gamma}{2}(t-t')} = \int_0^t dt'\,\langle\hat{\mathcal{F}}^\dagger(t)\,\hat{\mathcal{F}}(t')\rangle + \int_0^t dt'\,\langle\hat{\mathcal{F}}^\dagger(t')\,\hat{\mathcal{F}}(t)\rangle$, since $\langle\hat{\mathcal{F}}^\dagger(t_1)\,\hat{\mathcal{F}}(t_2)\rangle \propto \delta(t_1-t_2)$

By stationarity

$$= \int_0^t dt'\,\langle\hat{\mathcal{F}}^\dagger(\underbrace{t-t'}_{\equiv\tau})\,\hat{\mathcal{F}}(0)\rangle + \int_0^t dt'\,\langle\hat{\mathcal{F}}^\dagger(\underbrace{t'-t}_{\tau})\,\hat{\mathcal{F}}(0)\rangle = \int_{-t}^t d\tau\,\underbrace{\langle\hat{\mathcal{F}}^\dagger(\tau)\,\hat{\mathcal{F}}(0)\rangle}_{\Gamma\bar{n}\,\delta(\tau)} = \Gamma\bar{n} \qquad t>0$$

$$\Rightarrow \frac{d}{dt}\langle\hat{n}\rangle_R = -\Gamma\langle\hat{n}\rangle_R + \Gamma\bar{n}$$

$$\Rightarrow \boxed{\langle\hat{n}\rangle_R(t) = e^{-\Gamma t}\langle\hat{n}\rangle_R(0) + (1-e^{-\Gamma t})\,\bar{n}}$$

Average number of quanta in the oscillator thermalizes with the bath as expected.

Fluctuations Specta and the Quantum Regression "theorem"

In the theory of nonequilibrium dynamics, the "power spectrum" of system is given by the Fourier transform of the "auto correlation" function

$$S(\omega) = \int d\tau\, e^{i\omega\tau} \langle \hat{a}^\dagger(\tau)\, \hat{a}(0) \rangle \quad \text{(Weiner–Khinchine theorem)}$$

Consider thus, the two-time correlation function $\langle \hat{a}^\dagger(t)\hat{a}(t') \rangle_R$

$$\frac{d}{dt}\langle \hat{a}^\dagger(t)\, \hat{a}(t') \rangle_R = \left\langle \frac{d\hat{a}^\dagger}{dt} \hat{a}(t') \right\rangle_R = -\frac{\Gamma}{2}\langle \hat{a}^\dagger(t)\, \hat{a}(t') \rangle_R + \langle \hat{\mathcal{F}}^\dagger(t)\, \hat{a}(t') \rangle_R$$

If $t > t'$, in the Markov approximation, $\hat{a}(t')$ is not correlated with $\hat{\mathcal{F}}(t)$

$\Rightarrow \langle \hat{\mathcal{F}}^\dagger(t)\, \hat{a}(t') \rangle = 0$

$$\Rightarrow \boxed{\frac{d}{dt}\langle \hat{a}^\dagger(t)\, \hat{a}(t') \rangle = -\frac{\Gamma}{2}\langle \hat{a}^\dagger(t)\, \hat{a}(t') \rangle}$$

The two-time correlation function satisfies the same equation of motion as the mean. This is known as the "quantum regression theorem" as the fluctuations "regress" to the mean.

Generally, in the Markov approximation, given

$$\frac{d\hat{A}_i}{dt} = \sum_j \mathcal{L}^\dagger_{ij} \hat{A}_i + \hat{\mathcal{F}}_i(t),$$

adjoint Linblad matrix $\mathcal{L}^\dagger_{ij}$ including Hamiltonian, $\hat{\mathcal{F}}_i(t)$ = delta correlated noise

$$\Rightarrow \frac{d}{dt}\langle \hat{A}_i(t) \rangle_R = \sum_j \mathcal{L}^\dagger_{ij} \langle \hat{A}_j(t) \rangle \qquad \text{since } \langle \hat{\mathcal{F}}_i(0) \rangle = 0$$

Now consider $\langle \hat{A}_i(t+\tau)\, \hat{A}_k(t) \rangle_R$ with $\tau > t$

$$\frac{d}{d\tau}\langle \hat{A}_i(t+\tau)\, \hat{A}_k(\tau) \rangle_R = \sum_j \mathcal{L}_{ij} \langle \hat{A}_j(t+\tau)\, \hat{A}_k(\tau) \rangle + \langle \hat{\mathcal{F}}_i(\tau)\, \hat{A}_\ell(t) \rangle$$

In the Markov approximation, $\langle \hat{\mathcal{F}}_i(\tau)\, \hat{A}_\ell(t) \rangle = 0 \quad \tau > t$

$$\Rightarrow \boxed{\frac{d}{d\tau}\langle \hat{A}_i(t+\tau)\, \hat{A}_k(\tau) \rangle_R = \sum_j \mathcal{L}^\dagger_{ij} \langle \hat{A}_j(t+\tau)\, \hat{A}_k(t) \rangle \quad \tau > t}$$

Quantum regression theorem

For example, for the driven damped SHO in the rotation frame

$$\frac{d}{dt}\langle \hat{a}^\dagger(\tau)\, \hat{a}(0) \rangle = \left(-\Delta - \frac{i\Gamma}{2}\right)\langle \hat{a}^\dagger(\tau)\, \hat{a}(0) \rangle \Rightarrow \langle \hat{a}^\dagger(\tau)\, \hat{a}(0) \rangle = e^{-(\Delta - \frac{\Gamma}{2})\tau} \Rightarrow S(\omega) = \frac{1}{-i(\omega-\omega_0+\frac{\Gamma}{2})}$$

Lorenzians