Physics 581, Quantum Optics II
Problem Set #2
Due: Thursday February 12, 2026

Problem 1: Squeezing in the Heisenberg Picture (35 Points)

(a) An arbitrary Bogoliubov transformation on a single mode takes the form

$$\hat{b} = \hat{U}^\dagger \hat{a} \hat{U} = \mu \hat{a} - \nu \hat{a}^\dagger$$

Show that to preserve the commutation relations, $[\hat{b}, \hat{b}^\dagger] = 1 \implies |\mu|^2 - |\nu|^2 = 1$, and that the squeezing parameter $e^r = |\mu| + |\nu|$ and phase of squeezing $\theta = (\text{Arg}(\nu) + \text{Arg}(\mu))/2$.

Now consider a "pure squeezing" Hamiltonian, $\hat{H} = \hbar\kappa^* \hat{a}^2 + \hbar\kappa \hat{a}^{\dagger 2}$.

(b) Show that the Heisenberg equations of motion and solutions are

$$\frac{d\hat{a}}{dt} = -i2\kappa\hat{a}^\dagger, \ \frac{d\hat{a}^\dagger}{dt} = +i2\kappa^*\hat{a} \Rightarrow \hat{a}(t) = \cosh(r)\hat{a}(0) - e^{i2\theta}\sinh(r)\hat{a}^\dagger(0). \text{ Find } r, \theta.$$

This is the Bogoliubov transformation, corresponding to parametric amplification.
Note: $\hat{a}(t) = \mu\hat{a}(0) - \nu\hat{a}^\dagger(0)$, with $|\mu|^2 - |\nu|^2 = 1$.

(c) Now consider a squeezing interaction in the present of a rotation (caused by a phase mismatch), $\hat{H} = \hbar\Delta\hat{a}^\dagger\hat{a} + \hbar\kappa^*\hat{a}^2 + \hbar\kappa\hat{a}^{\dagger 2}$. Find the Heisenberg equations of motion and show that the solution is

$$\hat{a}(t) = \left[\cosh\Omega - i\frac{\Delta t}{\Omega}\sinh\Omega\right]\hat{a}(0) - \left[e^{i2\theta}\frac{r}{\Omega}\sinh\Omega\right]\hat{a}^\dagger(0)$$

where r, θ are as in part (b) and $\Omega = \sqrt{r^2 - (\Delta t)^2}$.

(d) Using this, show that this interaction leads to squeezing with squeezing parameter,

$$\tilde{r} = \sinh^{-1}\left(\frac{r}{\Omega}\sinh\Omega\right) = \log\left(\frac{r}{\Omega}\sinh\Omega + \sqrt{1 + \left(\frac{r}{\Omega}\sinh\Omega\right)^2}\right)$$

and show that only when we have perfect phase matching ($\Delta = 0$) do we achieve exponential growth with time (amplification) of one quadrature and de-amplifaction (squeezing) of the other.

Now consider a "shearing" Hamiltonian of the form $\hat{H} = \hbar\kappa\hat{X}^2$. This is an example of "non-phase matched" squeezing.

(e) For an initial vacuum state, qualitatively describe the evolution of the state in phase space. How does the "uncertainty bubble" evolve? Sketch this is phase space
(Hint: remember that X is the generator of displacements in P).

(f) Find the Heisenberg equations of motion for $\hat{X}$ and $\hat{P}$. Solve the differential equations and find the solution

$$\hat{a}(t) = \frac{\hat{X}(t) + i\hat{P}(t)}{\sqrt{2}}.$$

(h) Based on your solution (f) find the squeezing parameter and phase-quadrature of the squeezing and show that this agrees with the general solution in (c).