Physics 581, Quantum Optics II

Problem Set #4

Due: Monday March 12, 2026

Problem 1 and 2 required. Choose one of Problem 3 or 4. Do all four problems for extra credit.

Problem 1: Another form of the Wigner Function (15 points)

We have shown that Wigner function could be expressed as

$$W(\alpha) = \frac{1}{\pi} Tr\left(\hat{\rho}\hat{T}(\alpha)\right) = \frac{1}{\pi}\left\langle \hat{T}(\alpha)\right\rangle \text{, where } \hat{T}(\alpha) = \int \frac{d^2\beta}{\pi} \hat{D}(\beta) e^{\alpha\beta^* - \beta^*\alpha}$$

(a) Show that $\hat{T}(\alpha) = \hat{D}(\alpha)\hat{T}(0)\hat{D}^\dagger(\alpha)$.

(b) Show that $\hat{T}(0) = 2(-1)^{\hat{a}^\dagger\hat{a}}$. Hint: expand in the number basis and use the identities for Laguerre polynomials: $\langle m|\hat{D}(\alpha)|n\rangle = e^{-|\alpha|^2/2}\sqrt{\frac{n!}{m!}}\alpha^{m-n}\mathcal{L}_n^{(m-n)}(|\alpha|^2)$, $\int_0^\infty e^{-x/2}\mathcal{L}_n(x) = 2(-1)^n$

Note: the operator $(-1)^{\hat{a}^\dagger\hat{a}} = \sum_n (-1)^n |n\rangle\langle n| = \int dX |-X\rangle\langle X|$ is the "parity operator" (+1 for even parity, -1 for odd parity). Thus, we see that the Wigner function at the origin is given by the expected value of the parity.

$$W(0) = \frac{2}{\pi} Tr\left[\hat{\rho}(-1)^{\hat{a}^\dagger\hat{a}}\right] = \frac{2}{\pi}\sum_n (-1)^n \langle n|\hat{\rho}|n\rangle .$$

(c) Show that general expression

$$\hat{T}(\alpha) = 2\hat{D}(\alpha)(-1)^{\hat{a}^\dagger\hat{a}}\hat{D}^\dagger(\alpha) = 2\sum_n (-1)^n \hat{D}(\alpha)|n\rangle\langle n|\hat{D}^\dagger(\alpha) ,$$

$$\text{and thus } W(\alpha) = \frac{2}{\pi}\sum_n (-1)^n \langle n|\hat{D}^\dagger(\alpha)\hat{\rho}\hat{D}(\alpha)|n\rangle .$$

This expression provides a way to "measure" the Wigner function. One displaces the state to the point of interest, $\hat{D}^\dagger(\alpha)\hat{\rho}\hat{D}(\alpha)$, one then measures the photon statistics $p_{n\alpha} = \langle n|\hat{D}^\dagger(\alpha)\hat{\rho}\hat{D}(\alpha)|n\rangle$. Putting this in the parity sum gives $W(\alpha)$ at that point!

This is a form a quantum-state reconstruction, also known as "quantum tomography."

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Problem 2: Gaussian States in Quantum Optics (35 points)

The set of states whose quadrature fluctuations are Gaussian distributed about a mean value is an important class in quantum optics. These states have Gaussian Wigner functions. In this problem, we explore Gaussian states, their relationship to squeezing, and the canonical algebra of phase space.

Consider a field of *n-modes*, with quadrature defined by an ordered vector:

$$\mathbf{Z} = (X_1, P_1, X_2, P_2, \ldots, X_n, P_n).$$

The operators associated with these quadratures satisfy a set of canonical commutators relations that can be written compactly as,

$$\left[\hat{Z}_i, \hat{Z}_j\right] = i\,\Sigma_{ij}, \text{ where } \mathbf{\Sigma} = \bigoplus_{k=1}^{n} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ is a skew-symmetric matrix.}$$

We define an "inner product" in phase space as $\left(\mathbf{Z}|\mathbf{Q}\right) = Z_i \Sigma_{ij} Q_j$ (summed over repeated indices through this problem).

(a) Show that the phase space displacement operator can be written

$$\hat{D}(\mathbf{Z}) = \exp\left\{-i\left(\mathbf{Z}|\hat{\mathbf{Z}}\right)\right\}$$

A *Gaussian state* is one whose Wigner function is a Gaussian function on phase space. Recall the characteristic function of a quantum state is defined $\chi(\mathbf{Z}) = Tr\left(\hat{\rho}\hat{D}(\mathbf{Z})\right)$.
The general form of the characteristic function for a Gaussian state with is:

$$\chi(\mathbf{Z}) = \exp\left\{-\frac{1}{2}\left(\mathbf{Z}|\mathbf{C}|\mathbf{Z}\right) + i\left(\mathbf{d}|\mathbf{Z}\right)\right\}.$$

Where C_{ij} is known as the covariance matrix, and d_i is a real vector.

(b) Show that: $\left\langle \hat{Z}_i \right\rangle = d_i$, and $\frac{1}{2}\left\langle \Delta\hat{Z}_i\Delta\hat{Z}_j + \Delta\hat{Z}_j\Delta\hat{Z}_i \right\rangle = C_{ij}$, where $\Delta\hat{Z}_i \equiv \hat{Z}_i - \left\langle \hat{Z}_i \right\rangle$.
Hint: Recall how moments are found from the characteristic function.

The Gaussian state is thus determined by the mean position in phase space and the covariance of all the fluctuations.

(c) Find the Wigner function for a state with the general form of the characteristic function.

Let us restrict our attention to Gaussian states with zero mean (the mean is irrelevant to the statistics and can always be removed via a displacement operation). Consider now unitary transformations on the state. A particular class of transformations is the set that act as linear canonical transformations, i.e.

$$\hat{U}^\dagger \hat{Z}_i \hat{U} = S_{ij}\hat{Z}_j \text{, where } S_{ij} \text{ is a symplectic matrix, defined by } S^T \Sigma S = \Sigma .$$

A unitary map on the state transforms the state according to

$$\chi(\mathbf{Z}) \Rightarrow \chi'(\mathbf{Z}) = Tr\left(\hat{U}\hat{\rho}\hat{U}^\dagger \hat{D}(\mathbf{Z})\right) = Tr\left(\hat{\rho}\hat{U}^\dagger \hat{D}(\mathbf{Z})\hat{U}\right).$$

(d) Show that for a symplectic transformation, the characteristic function transforms as

$$\chi(\mathbf{Z}) \Rightarrow \chi(\mathbf{SZ})$$

and thus the action of the unitary is to *preserve the Gaussian statistics, by* transforming covariance matrix as $\mathbf{C} \Rightarrow \mathbf{S}^T\mathbf{CS}$.

(e) Show that the following operations preserve Gaussian statistics:

- Linear optics: $\hat{U} = \exp\left(-i\theta_{ij}\hat{a}_i^\dagger \hat{a}_j\right)$
- Squeezing: $\hat{U} = \exp\left(\zeta_{ij}^* \hat{a}_i \hat{a}_j - \zeta_{ij}\hat{a}_i^\dagger \hat{a}_j^\dagger\right)$

(f) For each of these, show how the covariance matrix of the Gaussian transforms.

(g) Starting with the vacuum (a Gaussian state) apply the squeezing operator above. Show that the symplectic transformation on the covariant matrix leads to the expected result.

Problem 3: Nonclassical light generation via the Kerr effect. (20 points)

In the classical (optical) Kerr effect, the index of refraction is proportional to the intensity. The quantum optical description is via the Hamiltonian,

$$\hat{H} = \frac{\hbar\kappa}{2} : \hat{n}^2 := \frac{\hbar\kappa}{2}\hat{a}^{\dagger 2}\hat{a}^2.$$

(a) Suppose the initial state is a coherent state $|\alpha_0\rangle$, with $|\alpha_0|^2 \gg 1$. At short times we can *linearize* this Hamiltonian about the mean field. This is good approximation when the fluctuation around the mean field are small. To do this we make the Mollow transformation $\hat{U}_{Mol} = e^{-i\kappa|\alpha_0|^2 t\hat{n}}\hat{D}(\alpha_0)$ so that $\hat{U}^{\dagger}_{Mol}\hat{a}\hat{U}_{Mol} = (\hat{a} + \alpha_0)e^{-i\kappa|\alpha_0|^2 t}$. Here $\hat{a}$ repress the quantum fluctuations around the mean-field. In mean-field frame resulting Hamiltonian is,

$\hat{H}_{MF} = \hat{U}^{\dagger}_{MF}\hat{H}\hat{U}_{MF} + i\dfrac{\partial \hat{U}^{\dagger}_{MF}}{\partial t}\hat{U}_{MF}$. Show that

$$\hat{H} \approx \hbar\kappa|\alpha_0|^2\hat{a}^{\dagger}\hat{a} + \frac{\hbar\kappa}{2}(\alpha_0^2\hat{a}^{\dagger 2} + \alpha_0^{*2}\hat{a}^2) = \frac{\hbar\kappa X_0^2}{2}\hat{X}^2,$$

$$\text{where } \alpha_0 = \frac{X_0}{\sqrt{2}} \text{ (chosen real)}, \hat{X} = \frac{\hat{a} + \hat{a}^{\dagger}}{\sqrt{2}}.$$

and that this leads to squeezing. Describe the resulting squeezed state.

(b) Now let's go beyond the linear approximation. For an arbitrary time, the time evolution operator is $\hat{U}(t) = e^{-i\hat{H}t/\hbar} = \sum_n e^{-i\kappa t n(n-1)/2}|n\rangle\langle n|$. At times $\kappa t_{N,M} = \frac{M}{N}2\pi$, (rational multiples of 2π), one can show (extra credit to prove)

$$\hat{U}(t_{N,M}) = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} e^{i\frac{M}{N}\pi k(k-1)}e^{i\frac{2\pi k}{N}\hat{n}} \text{ } (N \text{ even})$$

$$\hat{U}(t_{N,M}) = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} e^{i\frac{M}{N}\pi k^2}e^{i\frac{(2k+1)\pi}{N}\hat{n}} \text{ } (N \text{ odd})$$

Show that the state at these time is

$$|\psi(t_{N,M})\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{i\frac{M}{N}\pi k(k-1)} |e^{i\frac{2\pi k}{N}} \alpha_0\rangle \ (N \text{ even})$$

$$|\psi(t_{N,M})\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{i\frac{M}{N}\pi k^2} |e^{i\frac{(2k+1)\pi}{N}} \alpha_0\rangle \ \ (N \text{ odd})$$

Show that the state at the time $\kappa t_{2,1} = \pi$ is a "cat state."

(c) Show that the Wigner function evolves in time according to the following partial differential equation,

$$\frac{\partial W}{\partial t} = -i\kappa(|\alpha|^2 - 1)\left(\alpha^* \frac{\partial W}{\partial \alpha^*} - \alpha \frac{\partial W}{\partial \alpha}\right) - \frac{i\kappa}{4}\left(\alpha^* \frac{\partial}{\partial \alpha} - \alpha \frac{\partial}{\partial \alpha^*}\right) \frac{\partial^2 W}{\partial \alpha \partial \alpha^*}.$$

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The first term is the classical dynamics from the Poison bracket— interpret these dynamics physically.

(d) Using a decomposition of the state in the Fock basis $|\psi(t)\rangle = \sum_n c_n(t)|n\rangle$,

show that that Wigner function as a function of time is

$$W(\alpha, t) = \sum_n^{\infty} |c_n(t)|^2 W_{n,n}(\alpha) + \left(\sum_{n,m>n}^{\infty} c_n(t) c_m^*(t) W_{n,m} + c.c. \right)$$

$$\text{where } W_{n,m} = \frac{2}{\pi}(-1)^n \sqrt{\frac{n!}{m!}} (2\alpha^*)^{m-n} \mathscr{L}_n^{m-n}(4|\alpha|^2) e^{-2|\alpha|^2}.$$

(e) 5 pts extra credit: Use (d) to make a movie of $W(\alpha, t)$ for $\alpha_0 = 4$ (you will need to appropriately truncate the Fock space). Comment on you result.

Problem 4: The Bargmann and Stellar Representations

Another important phase space representation, closely related to the Husimi representation, is the *Bargmann representation* (also known as the Segal-Bargmann representation). Bargmann was an applied mathematician you considered the following problem. Consider the Hilbert space of a single mode (equivalent to wave functions in the real line). Given a state $|\psi\rangle$, we seek a representation on the complex plane $\psi_B(\alpha)$ such $\hat{a}|\psi\rangle \Rightarrow \frac{\partial}{\partial\alpha}\psi_B(\alpha)$, $\hat{a}^\dagger|\psi\rangle \Rightarrow \alpha\,\psi_B(\alpha)$. This is a good representation since $[\hat{a},\hat{a}^\dagger] = [\frac{\partial}{\partial\alpha},\alpha] = 1$. A standard coherent state $|\alpha\rangle = \hat{D}(\alpha)|0\rangle = e^{-\frac{|\alpha|^2}{2}}e^{\alpha\hat{a}^\dagger}|0\rangle$.

Define the (unnormalized) "Bargmann coherent state" as $\|\alpha\rangle \equiv e^{\alpha\hat{a}^\dagger}|0\rangle$.

(a) Show that $\hat{a}\|\alpha\rangle = \alpha\|\alpha\rangle$, and $\hat{a}^\dagger\|\alpha\rangle = \frac{\partial}{\partial\alpha}\|\alpha\rangle$, and that thus that $\langle\alpha^*\|\hat{a}|\psi\rangle = \frac{\partial}{\partial\alpha}\psi_B(\alpha)$, $\langle\alpha^*\|\hat{a}^\dagger|\psi\rangle = \alpha\,\psi_B(\alpha)$, where $\psi_B(\alpha) \equiv \langle\alpha^*\|\psi\rangle = e^{\frac{|\alpha|^2}{2}}\langle\alpha^*|\psi\rangle$ the Bargmann representation.

(b) For an arbitrary pure state, expanded in the Fock basis, $|\psi\rangle = \sum_n c_n|n\rangle$, show that $\psi_B(\alpha) = \sum_n \frac{c_n}{n!}\alpha^n$. The Bargmann representation is thus *everywhere analytic* on the complex plane.

(c) Show that the Husimi representation of a pure that is $Q(\alpha) = \frac{e^{-|\alpha|^2}}{\pi}|\psi_B(\alpha^*)|^2$.

The zeros of the the Husimi function are thus the zeros of the Bargmann representation. This intimately related to the nodes of the Wigner function, as the Q function will have a zero whenever *W* goes from positive to negative (See Lükenhaus and Barnett,PRA **51**, 3340 (1995).

(d) The complex function that is everywhere analytic on the complex plane can be factorized according to the Weierstrass-Hadamard theorem. For the Bargmann function one can show that this takes the form,

$$\psi_B(\alpha) = \mathcal{N}\prod_{i=1}^{r*}\left(\alpha - \beta_i\right)G(\alpha)$$

where $\mathcal{N}$ is a normalization constant, β_i is a zero, r_* is the "stellar rank," including degeneracy, and $G(\alpha)$ is a Gaussian function. Using this show that

$$|\psi\rangle = \psi_B(\hat{a}^\dagger) = \mathcal{N}\prod_{i=1}^{r*}\hat{D}(\beta_i)\hat{a}^\dagger\hat{D}^\dagger(\beta_i)|G_\psi\rangle.$$

where $|G_\psi\rangle$ is a Gaussian state. This says that an arbitrary state of the mode can be constructed from a base Gaussian state followed by a series of displaced phon additions determined by the stellar rank.