

Physics 581, Quantum Optics II

Problem Set #5

Due: Tuesday March 31, 2026

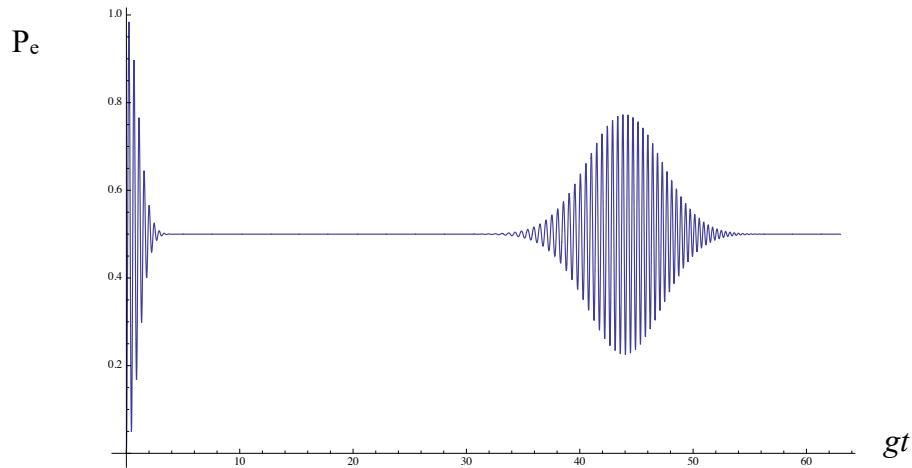
Problem 1: Entanglement and the Jaynes-Cummings Model (30 points)

One of the most fundamental paradigms in quantum optics is the coupling of a two-level atom to a single mode of the quantized electromagnetic field. In the rotating wave approximation, this is governed by the Jaynes-Cummings model (JCM),

$$\hat{H} = \hbar\omega_c \hat{a}^\dagger \hat{a} + \hbar\omega_0 \frac{\hat{\sigma}_z}{2} + \hbar g (\hat{\sigma}_+ \hat{a} + \hat{a}^\dagger \hat{\sigma}_-).$$

This is a bipartite system with tensor product Hilbert space for the atom and field, $\mathcal{H}_{AF} = \mathcal{h}_A \otimes \mathcal{h}_F$, where \mathcal{h}_A is the two-dimensional Hilbert space of the two-level atom, and \mathcal{h}_F is the infinite dimensional Hilbert space of the harmonic oscillator that describes the mode. The goal of this problem is to understand the entanglement between the atom and mode, generated by the JCM.

Last semester, we studied how this leads to collapse and revival of Rabi oscillation that follows from an initial product state with the field in a coherent state and the atom in, e.g., the ground state $|\Psi(0)\rangle_{AF} = |g\rangle_A \otimes |\alpha\rangle_F$. The probability to find the atom in the excited state oscillates as shown (here for $\langle n \rangle = |\alpha|^2 = 49$)



The collapse is due to the variation of the quantum Rabi oscillations with different number; the revival is uniquely a quantum effect arising from the discreteness of the quantized field, occurring at a time $gt_r \approx 2\pi\sqrt{\langle n \rangle}$ for large $\langle n \rangle$.

(a) Let us consider the case on resonance with $\omega_c = \omega_0$. Show that in the interaction picture, the state at time t the joint state takes the form

$$|\Psi(t)\rangle_{AF} = |g\rangle_A \otimes |C(t)\rangle_F + |e\rangle_A \otimes |S(t)\rangle_F$$

where $|C(t)\rangle_F = \sum_{n=0}^{\infty} c_n \cos(\sqrt{n}gt)|n\rangle$, $|S(t)\rangle_F = -i \sum_{n=0}^{\infty} c_{n+1} \sin(\sqrt{n+1}gt)|n\rangle$, $c_n = (\alpha^n / \sqrt{n!}) e^{-|\alpha|^2/2}$.

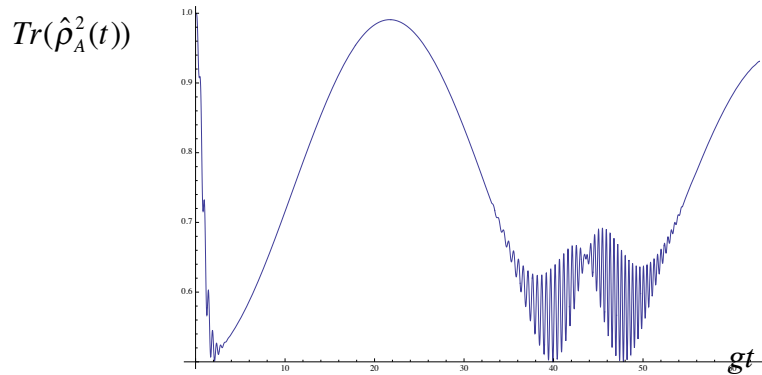
Note $|C(t)\rangle_F, |S(t)\rangle_F$ are not normalized, nor are they orthogonal.

(b) Show that the marginal state of the atom in the $\{|g\rangle, |e\rangle\}$ basis is

$$\hat{\rho}_A(t) = \begin{bmatrix} \langle C(t)|C(t)\rangle & \langle S(t)|C(t)\rangle \\ \langle C(t)|S(t)\rangle & \langle S(t)|S(t)\rangle \end{bmatrix} = \frac{1}{2}(\hat{1} + \vec{Q}(t) \cdot \hat{\sigma}).$$

Write an expression for Bloch vector $\vec{Q}(t)$.

(c) Write the purity of the marginal (a measure of the entanglement between the atom and field), in terms of the Bloch vector. Numerically calculate this and plot as a function of time for $\langle n \rangle = |\alpha|^2 = 49$. Your graph should look like



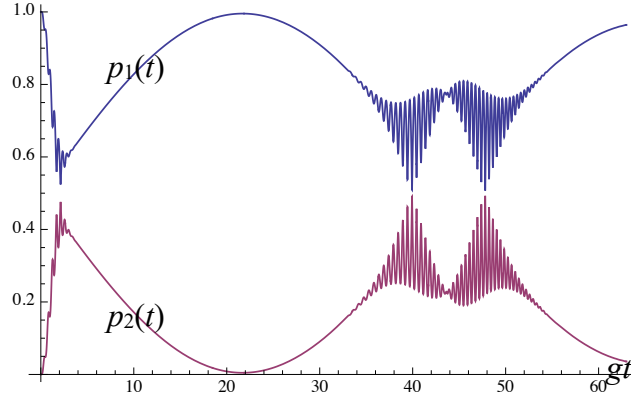
This plot shows a few surprising features. During the collapse the atom and field become highly entangled, as indicated by the rapid decrease in the atomic purity. However, at half the revival time, $gt_r / 2 \approx \pi\sqrt{\langle n \rangle}$, when the inversion looks to be flat, the purity returns to near unity, indicating that the atom and field become *separable*. The atom and field then become re-entangled. When the Rabi oscillations once again revive, the purity again increases, but now to a value near to unity. Our goal now is to use the Schmidt decomposition to understand this.

(d) Given the initial pure state of the joint system and the unitary evolution according to the JCM, we know that at all times we can express the state in terms of Schmidt decomposition.

$$|\Psi(t)\rangle_{AF} = \sum_{\mu=1}^2 \sqrt{p_{\mu}(t)} |u_{\mu}(t)\rangle_A \otimes |v_{\mu}(t)\rangle_F.$$

Note, even though the field mode is infinite dimensional, the maximum Schmidt number is 2.

Express the two values of $p_\mu(t)$ in terms of the Bloch vector $\vec{Q}(t)$. Calculate numerically and plot as function of time. Your graphs should look like the following:



Comment on this and what it means for the entanglement.

(e) We can find the Schmidt vectors by the following procedure.

- Find the atomic Schmidt vectors $\{|u_\mu(t)\rangle_A\}$ as the eigenvectors of the marginal state $\hat{\rho}_A(t)$ in the standard basis $\{|g\rangle, |e\rangle\}$.

- Using $|\Psi(t)\rangle_{AF} = |g\rangle_A \otimes |C(t)\rangle_F + |e\rangle_A \otimes |S(t)\rangle_F = \sum_{\mu=1}^2 \sqrt{p_\mu(t)} |u_\mu(t)\rangle_A \otimes |v_\mu(t)\rangle_F$, find an expression for the two Schmidt vectors of the field $\{|v_\mu(t)\rangle_F\}$ in terms of $|C(t)\rangle_F, |S(t)\rangle_F, p_\mu(t)$.

(f) We can see the (approximate) separation between atom and field at half the revival time for large $\langle n \rangle$ as follows. Show that in this limit,

$$g\sqrt{n+1}t_r/2 \approx g\sqrt{nt_r}/2 + \pi/2, \quad c_{n+1} \approx e^{+i\phi} c_n, \quad \text{where } c_n = (\alpha^n / \sqrt{n!}) e^{-|\alpha|^2/2} \text{ and } \alpha = \sqrt{\langle n \rangle} e^{i\phi}.$$

Using this, show that

$$|\Psi(t_r/2)\rangle_{AF} \approx (|g\rangle_A - ie^{-i\phi}|e\rangle_A) \otimes |C(t_r/2)\rangle_F.$$

Thus, we see that the system is separable, with the atom in an equal superposition depending on the phase of the coherence state.

(g) Extra credit (5 points): More generally show that if $|\Psi(0)\rangle_{AF} = (a|g\rangle_A + b|e\rangle_A) \otimes |\alpha\rangle_F$

$$|\Psi(t_r/2)\rangle_{AF} \approx (|g\rangle_A - ie^{-i\phi}|e\rangle_A) \otimes (a|C(t_r/2)\rangle_F + b|S(t_r/2)\rangle_F)$$

This result shows that *regardless of the atomic initial state*, at half the revival time, the atom goes to the same state. The information about the initial atomic superposition is transferred to the field in a kind of “swap gate.”

For large a , the two field states are macroscopically distinguishable. This is kind of “Schrödinger cat”.

Problem 2: The EPR state (35 points)

The Einstein-Podolsky-Rosen (EPR) paradox is based around a thought experiment of measurements on an entangled state of the motion of two particles. The EPR state is a simultaneous eigenstate of relative position and the center-of-mass momentum

$$\left(\hat{X}_A - \hat{X}_B\right)|EPR\rangle = X_{rel}|EPR\rangle, \quad \left(\hat{P}_A + \hat{P}_B\right)|EPR\rangle = P_{com}|EPR\rangle$$

The purpose of this problem is to show how one can create an approximation to this state in quantum optics, and to study their entanglement properties.

Consider now a parametric oscillator beyond the perturbative limit, where two modes (A and B) are phase matched with the pump. The resulting output state is a two-mode squeezed vacuum state

$$|0,0\rangle_r = \hat{S}_{AB}(r)|0\rangle_A|0\rangle_B = e^{r(\hat{a}^\dagger\hat{b}^\dagger - \hat{a}\hat{b})}|0\rangle_A|0\rangle_B$$

Our goal is to show that in the limit of infinite squeezing, this is the EPR state.

(a) Show that, $\hat{S}_{AB}^\dagger(\hat{P}_A \pm \hat{P}_B)\hat{S}_{AB} = (\hat{P}_A \pm \hat{P}_B)e^{\mp r}$, and thus this operation squeezes the “relative position” and “center-of-mass momentum” quadratures.

(b) Show that $\hat{S}_{AB}^\dagger\hat{X}_A\hat{S}_{AB} = \cosh r\hat{X}_A + \sinh r\hat{X}_B$, $\hat{S}_{AB}^\dagger\hat{X}_B\hat{S}_{AB} = \cosh r\hat{X}_B + \sinh r\hat{X}_A$. This is a Heisenberg statement. (d) From this argue that, up to normalization (which is tricky)

$$\hat{S}_{AB}(r)|X_A\rangle_A|X_B\rangle_B = |\cosh rX_A + \sinh rX_B\rangle_A|\cosh rX_B + \sinh rX_A\rangle_B$$

(c) Show that the (normalized) position space wave function for the two modes is

$$\Psi_r(X_A, X_B) = \langle X_A|\langle X_B||0,0\rangle_r = \frac{1}{\sqrt{\pi}} e^{-\frac{(X_A - X_B)^2}{4e^{-2r}}} e^{-\frac{(X_A + X_B)^2}{4e^{+2r}}} \quad (\text{plot for } r=2)$$

and in the limit of infinite squeezing $\lim_{r \rightarrow \infty} \Psi_r(X_A, X_B) \Rightarrow \delta(X_A - X_B)$

(d) By similar arguments, show that the (normalized) momentum space wave function is

$$\tilde{\Psi}_r(P_A, P_B) = \langle P_A|\langle P_B||0,0\rangle_r = \frac{1}{\sqrt{\pi}} e^{-\frac{(P_A + P_B)^2}{4e^{-2r}}} e^{-\frac{(P_A - P_B)^2}{4e^{+2r}}} \quad (\text{plot for } r=2)$$

and in the limit of infinite squeezing $\lim_{r \rightarrow \infty} \tilde{\Psi}_r(P_A, P_B) \Rightarrow \delta(P_A + P_B)$

Thus argue that in the limit of infinite squeezing, the two-mode squeezed vacuum is the EPR state.

(f) Show that in the limit of infinite squeezing, the two-mode squeezed state can be expressed as

$$\lim_{r \rightarrow \infty} |0,0\rangle_r \Rightarrow |EPR\rangle = \int dX |X\rangle_A \otimes |X\rangle_B = \int dP |P\rangle_A \otimes |-P\rangle_B = \sum_n |n\rangle_A \otimes |n\rangle_B$$

Note: This is maximally entangled state in infinite dimensions. It is not a physical state, however, as it requires infinite energy. Nonetheless, we approximate it with large, but finite squeezing.

(g) Show that the Wigner function for the two-mode state is

$$W(X_A, P_A, X_B, P_B) = |\Psi_r(X_A, X_B)|^2 |\tilde{\Psi}_r(P_A, P_B)|^2 = \frac{1}{\pi^2} e^{-\frac{(X_A - X_B)^2 + (P_A + P_B)^2}{2e^{-2r}}} e^{-\frac{(X_A + X_B)^2 + (P_A - P_B)^2}{2e^{+2r}}}$$

(h) The Wigner function is positive, meaning there is a classical local probabilistic description of joint measurements of X_A, X_B, P_A, P_B . What are the implications for the EPR paradox and Bell's inequalities?