

Physics 581, Open Quantum Systems

Problem Set #7 EXTRA CREDIT

Due: Tuesday April 28, 2026

Problem 1: The transfer of coherences in the decay of an oscillator (20 points)

In the simple harmonic oscillator, the decay rate of population in a Fock state $|n\rangle$ is proportion to n . This makes sense physically, e.g., for a bosonic modes, as the probability of losing a photon will depend on the number of photons in the mode. However, for a coherent state, the rate of decay is independent of the amplitude! We understand this through the “transfer of coherence.” Because of the equal spacing of the energy levels, *coherence fed in the master equation*. This transfer of coherences explains the way in which the mean amplitude decays. Let’s examine this more closely.

(a) Consider the damped simple harmonic oscillator at finite temperature. Find the equation of motion of the off-diagonal elements (coherences) $\dot{\rho}_{n+1,n}$.

(b) Given

$$\langle \hat{a} \rangle = Tr(\hat{a}\hat{\rho}) = \sum_{n=0}^{\infty} \sqrt{n+1} \rho_{n+1,n} \quad \text{show that}$$

$$\frac{d}{dt} \langle \hat{a} \rangle = \sum_{n=0}^{\infty} \sqrt{n+1} \dot{\rho}_{n+1,n} = \left(-i\omega - \frac{\Gamma}{2}\right) \langle \hat{a} \rangle,$$

independent of the temperature of the bath.

(c) Now consider the *nonlinear* Kerr oscillator, with Hamiltonian

$$\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\kappa\hat{a}^{\dagger 2}\hat{a}^2$$

Including the same Lindbladian for the damped SHO at zero temperature, find the equation of motion of $\dot{\rho}_{n+1,n}$, and from this show by the same procedure in (b)

$$\frac{d}{dt} \langle \hat{a} \rangle = \left(-i\omega - \frac{\Gamma}{2}\right) \langle \hat{a} \rangle - i\kappa \langle \hat{n}\hat{a} \rangle$$

(d) Repeat this calculation by directly finding $d\langle \hat{a} \rangle/dt$ from the commutators.

Thus, for the nonlinear oscillator the transfer of coherences is not perfect as the spacing between levels is not equal. Thus, the decay of the oscillator depends on its amplitude.

(e) But wait a minute! How did we get away with the same Lindbladian? Why are any transfer of coherences included in the first place, since the spacing between levels is not equal. Why didn’t we need to include the Kerr Hamiltonian in the derivation of the Lindbladian?

Problem 2: Decoherence and cat states (35 points)

Consider the damped simple harmonic oscillator governed by the master equation

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] - \frac{\Gamma}{2}(\hat{a}^\dagger \hat{a} \rho + \rho \hat{a}^\dagger \hat{a}) + \Gamma \hat{a} \rho \hat{a}^\dagger, \text{ where } \hat{H} = \hbar\omega \hat{a}^\dagger \hat{a}.$$

(a) Suppose we prepare an initial pure coherent state, $\hat{\rho}(0) = |\alpha\rangle\langle\alpha|$. From the master equation,

$$\hat{\rho}(dt) = \left(\hat{1} - \frac{i}{\hbar} \hat{H}_{\text{eff}} dt \right) \hat{\rho}(0) \left(\hat{1} + \frac{i}{\hbar} \hat{H}_{\text{eff}}^\dagger dt \right) + \Gamma dt \hat{a} \hat{\rho}(0) \hat{a}^\dagger \quad \hat{H}_{\text{eff}} = \hat{H} - i \frac{\hbar\Gamma}{2} \hat{a}^\dagger \hat{a}$$

Show that $\hat{\rho}(dt) = |\alpha e^{-i\omega dt} e^{-\Gamma dt/2}\rangle\langle\alpha e^{-i\omega dt} e^{-\Gamma dt/2}|$ (Hint keep terms only to order dt). From this argue that for finite time $\hat{\rho}(t) = |\alpha(t)\rangle\langle\alpha(t)|$, where $\alpha(t) = \alpha e^{-i\omega t} e^{-\Gamma t/2}$, i.e., the state remains a pure coherent state for all times, with a complex amplitude that follows the classical trajectory.

(b) Now consider an initial pure state which is a superposition of coherent states $|\psi(0)\rangle = \mathcal{N}(|\alpha\rangle + |\beta\rangle)$, where $\mathcal{N}^{-2} = 2(1 + \text{Re}\langle\alpha|\beta\rangle)$ is the normalization. Consider the different map generated by the master equation. Show that,

$$\begin{aligned} \mathcal{E}(dt)[|\alpha\rangle\langle\beta|] &= \left(\hat{1} - \frac{i}{\hbar} \hat{H}_{\text{eff}} dt \right) |\alpha\rangle\langle\beta| \left(\hat{1} + \frac{i}{\hbar} \hat{H}_{\text{eff}}^\dagger dt \right) + \Gamma dt \hat{a} |\alpha\rangle\langle\beta| \hat{a}^\dagger \\ &= \langle\beta|\alpha\rangle^{\Gamma dt} |\alpha e^{-i\omega dt} e^{-\Gamma dt/2}\rangle\langle\beta e^{-i\omega dt} e^{-\Gamma dt/2}| \end{aligned}$$

(c) From this show that for an initial superposition of coherent states, the solution to the master equation is

$$\begin{aligned} \hat{\rho}(t) &= \mathcal{N}^2 (|\alpha(t)\rangle\langle\alpha(t)| + f(t)|\alpha(t)\rangle\langle\beta(t)| + f^*(t)|\beta(t)\rangle\langle\alpha(t)| + |\beta(t)\rangle\langle\beta(t)|) \\ \alpha(t) &= \alpha e^{-i\omega t} e^{-\Gamma t/2}, \quad \beta(t) = \beta e^{-i\omega t} e^{-\Gamma t/2}, \quad f(t) = \langle\beta|\alpha\rangle^{(1-e^{-\Gamma t})}. \end{aligned}$$

(d) Given the result (c) write the Wigner function for the state as a function time.

(e) For a Schrödinger cat state with $\beta = -\alpha$ and $|\alpha| \gg 1$, qualitatively describe the time evolution of the state.

(f) (Extra credit 10 points). Make a movie of the time evolution of the cat state evolving according to master equation for three cases: $\alpha=1,4,10$. For simplicity, go into the rotating frame and by setting $\omega=0$. Comment on your results.