Short Course in Quantum Information Lecture 6

Decoherence, Errors, Error Correction

Course Info

- All materials downloadable @ website

http://info.phys.unm.edu/~deutschgroup/DeutschClasses.html

- Syllabus

Lecture 1: Intro
Lecture 2: Formal Structure of Quantum Mechanics
Lecture 3: Entanglement
Lecture 4: Qubits and Quantum Circuits
Lecture 5: Algorithms
Lecture 6: Decoherence and Errors
Lecture 7: Quantum Cryptography
Lecture 8: Physical Implementations

Three Main Quantum Algorithms

- **Shor's Algorithm (Quantum Fourier Transform)**
  - O(n^2log$n$) for an number an n-bit number.
  - Generalizations: "Hidden subgroup".
- **Grover's Algorithm (Unstructured database search)**
  - O($N^{1/2}$) for a database with N *entries*: Provably optimal.
  - Precision measurement.
- **Quantum Simulations (Solving Schrödinger's equation)**
  - Properties of many body quantum systems.
  - "Analog" quantum computer.

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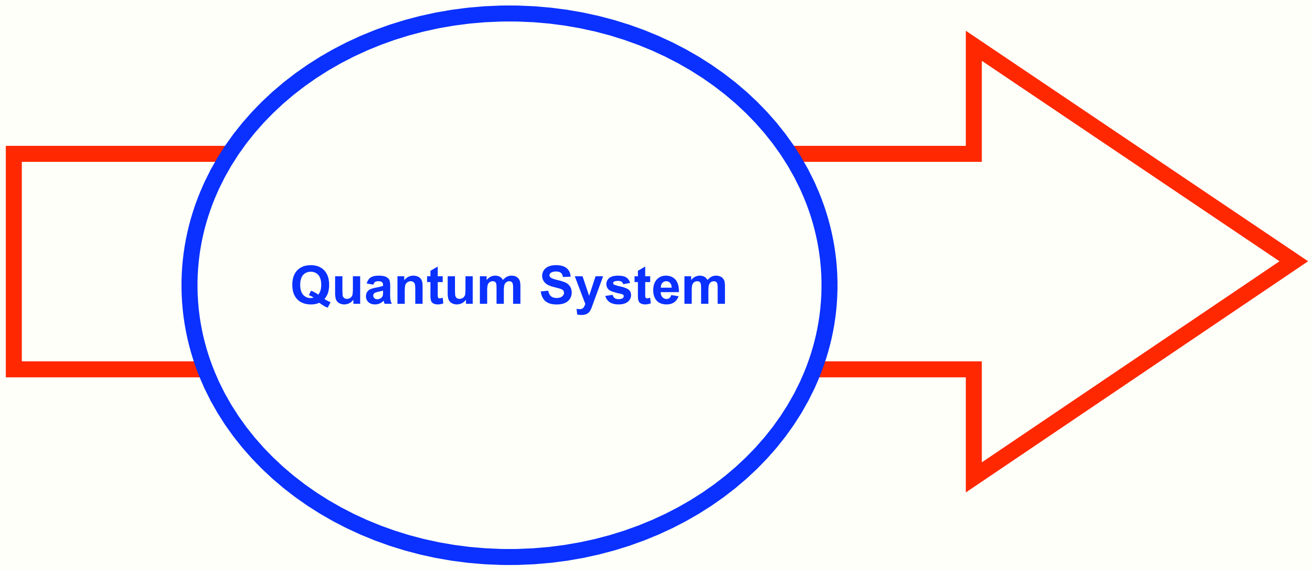
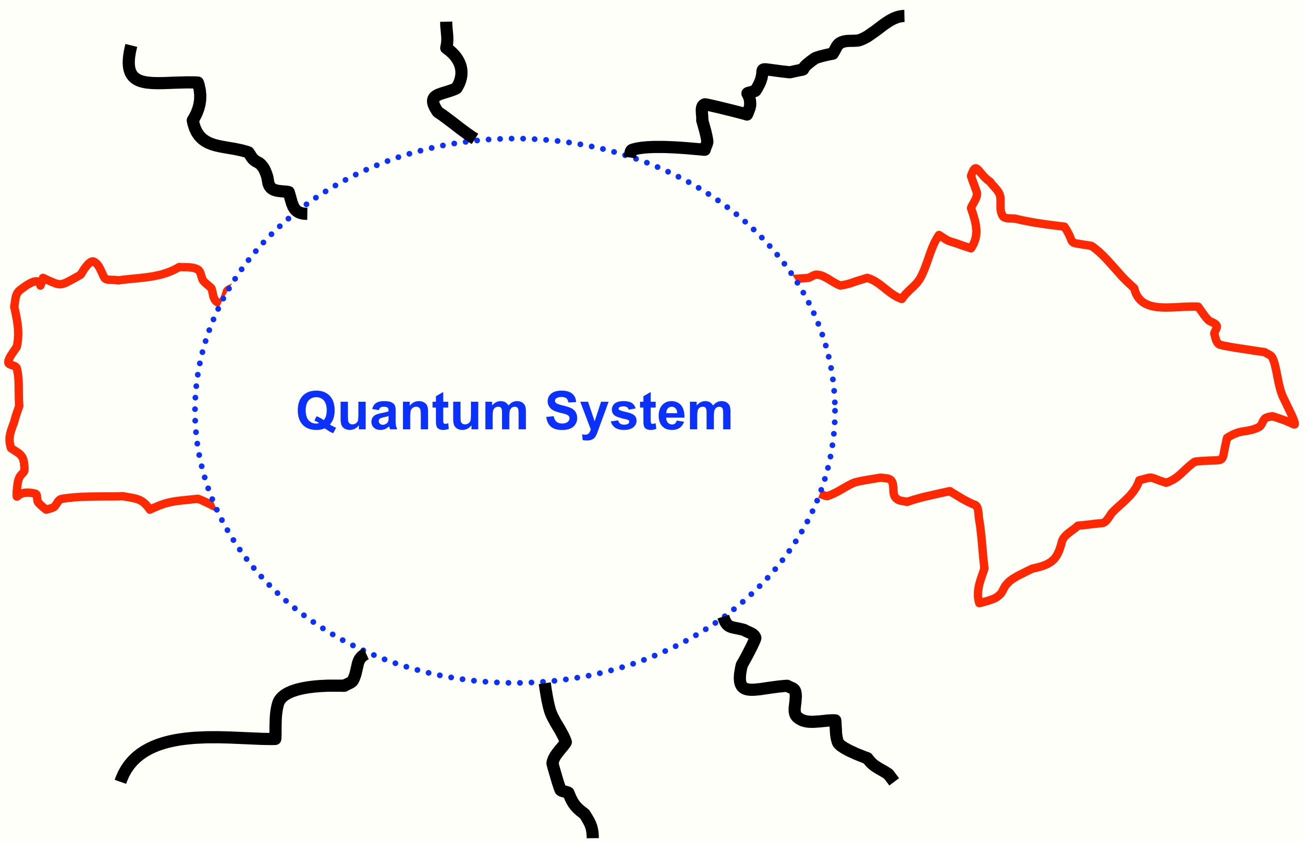
Quantum Mechanics: Ideal Picture



Perfectly controlled *closed* quantum system

Quantum Mechanics in the Real World

Noisy controls and coupling to the "environment"



Quantum mechanics in ***open*** quantum systems

Review: Coherent Superpositions

Pure State of a Qubit $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$

$$p_a = \left|\langle a|\psi\rangle\right|^2 = \left|c_0\langle a|0\rangle + c_1\langle a|1\rangle\right|^2$$

$$= \underbrace{|c_0|^2\,\left|\langle a|0\rangle\right|^2}_{p_0 p_{a|0}} + \underbrace{|c_1|^2\,\left|\langle a|1\rangle\right|^2}_{p_1 p_{a|1}} + \underbrace{c_0 c_1^*\langle a|0\rangle\langle 1|a\rangle + c_1 c_0^*\langle a|1\rangle\langle 0|a\rangle}_{Interference}$$

Quantum interference between 0 and 1 governed by

$$c_0 c_1^* = |c_0|\,|c_1|\exp\left[i(\phi_0 - \phi_1)\right]$$

Well defined phase difference → ***Coherence***

From Qubits to Bits

Qubit pure state $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$

Open quantum system: Decoherence

Classical probabilistic bit $\{p_0, |0\rangle;\ \ p_1, |1\rangle\}$

Incoherent Statistical Mixture

Statistical Mixture $\{p_0, |0\rangle;\ p_1, |1\rangle\}$

$$p_a == \underbrace{p_0 \left|\langle a|0\rangle\right|^2}_{p_0 p_{a|0}} + \underbrace{p_1 \left|\langle a|1\rangle\right|^2}_{p_1 p_{a|1}}$$

No Interference

Some White Lies

Quantum states are vectors in Hilbert Space.

Quantum states are density operators.

Quantum dynamics are unitary maps.

Quantum dynamics are completely positive maps.

Measurements are projectors onto orthogonal subspaces.

Measurements are "POVMS".

Density Operators

Consider again pure state: $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$

$$p_a = |c_0|^2 \langle a|0\rangle\langle 0|a\rangle + |c_1|^2 \langle a|1\rangle\langle 1|a\rangle + c_0 c_1^* \langle a|0\rangle\langle 1|a\rangle + c_1 c_0^* \langle a|1\rangle\langle 0|a\rangle$$

$$p_a = \langle a|\hat{\rho}|a\rangle$$

Density operator for a pure state:

$$\hat{\rho} = |c_0|^2 |0\rangle\langle 0| + |c_1|^2 |1\rangle\langle 1| + c_0 c_1^* |0\rangle\langle 1| + c_1 c_0^* |1\rangle\langle 0|$$

$$= \begin{bmatrix} |c_0|^2 & c_0 c_1^* \\ c_1 c_0^* & |c_1|^2 \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} \begin{bmatrix} c_0^* & c_1^* \end{bmatrix} = |\psi\rangle\langle\psi|$$

From pure states to mixed states

- **Unknown preparation procedure (e.g. thermal state)**

$$\left\{ p_i, |\psi_i\rangle \right\} \Rightarrow \hat{\rho} = \sum_{i=1}^{N} p_i |\psi_i\rangle\langle\psi_i|$$

$$\hat{\rho} = \sum_i p_i \begin{bmatrix} |c_0^i|^2 & c_0^i c_1^{i*} \\ c_1^i c_0^{i*} & |c_1^i|^2 \end{bmatrix} = \begin{bmatrix} \overline{|c_0|^2} & \overline{c_0 c_1^*} \\ \overline{c_1 c_0^*} & \overline{|c_1|^2} \end{bmatrix}$$

Statistic mixture of states can wash out interference

Decoherence

From pure states to mixed states

- **Noise in control fields**

Hamiltonian parameterized by control pulses: $\hat{H}\left(\lambda(t)\right)$

$$\hat{\rho}(t) = \sum_{\lambda} p_{\lambda} \begin{bmatrix} \left|c_0^{\lambda}(t)\right|^2 & c_0^{\lambda}(t) c_1^{\lambda *} \\ c_1^{\lambda}(t) c_0^{\lambda *}(t) & \left|c_1^{\lambda}(t)\right|^2 \end{bmatrix} = \begin{bmatrix} \overline{\left|c_0\right|^2}(t) & \overline{c_0 c_1^*}(t) \\ \overline{c_1 c_0^*}(t) & \overline{\left|c_1\right|^2}(t) \end{bmatrix}$$

Exponential decay processes:

Population relaxation: T_1

Dephasing rate: T_2

From pure states to mixed states

- **Entanglement with the environment**

Consider bipartite system two qubits: $\left|\Psi\right\rangle_{AB}$

Joint Probability Distribution $P_{AB}(i,j) = \left|\left\langle i,j \middle| \Psi_{AB} \right\rangle\right|^2$

Marginal Probability Distributions

$$P_A(i) = \sum_j P_{AB}(i,j)$$

$$P_B(j) = \sum_i P_{AB}(i,j)$$

What is the state of the individual qubits?

From pure states to mixed states

- **Entanglement with the environment**

Consider bipartite Bell state: $\left|\Phi^{(+)}\right\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B\right)$

$$P_{AB}(0,0) = P_{AB}(1,1) = 1/2 \qquad P_{AB}(0,1) = P_{AB}(0,1) = 0$$

$$\Rightarrow \quad \begin{array}{l} P_A(0) = P_A(1) = 1/2 \\ P_B(0) = P_B(1) = 1/2 \end{array}$$

$$\hat{\rho}_A = \begin{bmatrix} 1/2 & ?? \\ ?? & 1/2 \end{bmatrix} \qquad \hat{\rho}_B = \begin{bmatrix} 1/2 & ?? \\ ?? & 1/2 \end{bmatrix}$$

From pure states to mixed states

$$\left|\Phi^{(+)}\right\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B\right)$$

Consider probability distribution in X-basis: $|\pm\rangle \equiv \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right)$

$$P_{AB}(+,+) = P_{AB}(-,-) = 1/2 \qquad P_{AB}(+,-) = P_{AB}(-,+) = 0$$

$$\Rightarrow \quad P_A(+) = P_A(-) = 1/2$$
$$P_B(+) = P_B(-) = 1/2$$

True in *any basis.*

$$\Rightarrow \quad \hat{\rho}_A = \hat{\rho}_B = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$ Complete mixed state

From pure states to mixed states

Lessons:

1. Given a pure ***entangled*** state of the joint system of particles, e.g. two qubits, the state of the subsystems is ***mixed.***

The sum is greater than its parts.

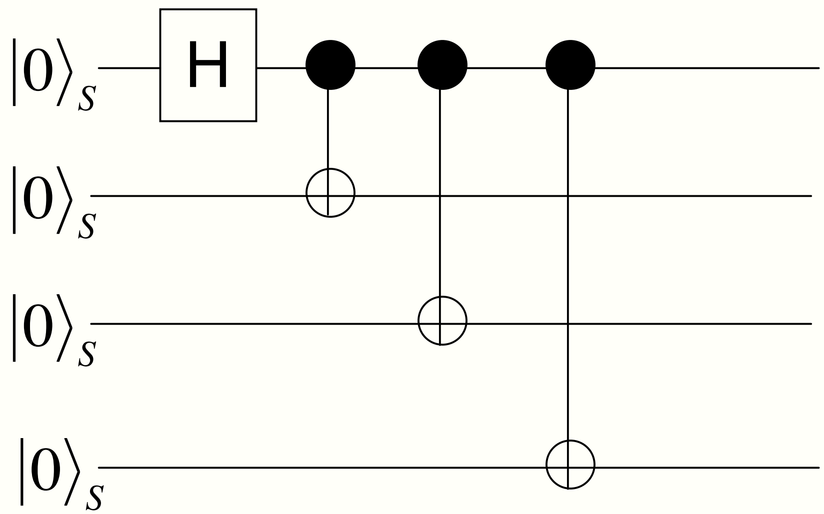
2. Entanglement of the "system" degrees of freedom with the environment leads to ***decoherence*** of the system.

3. The environment can store a "record" of the state of the system thus making the alternatives ***in-principle distinguishable.***

$$\left|\Phi^{(+)}\right\rangle_{SE} = \frac{1}{\sqrt{2}}\left(|0\rangle_S \otimes |0\rangle_E + |1\rangle_S \otimes |1\rangle_E\right)$$

Where are the Schrödinger Cats?

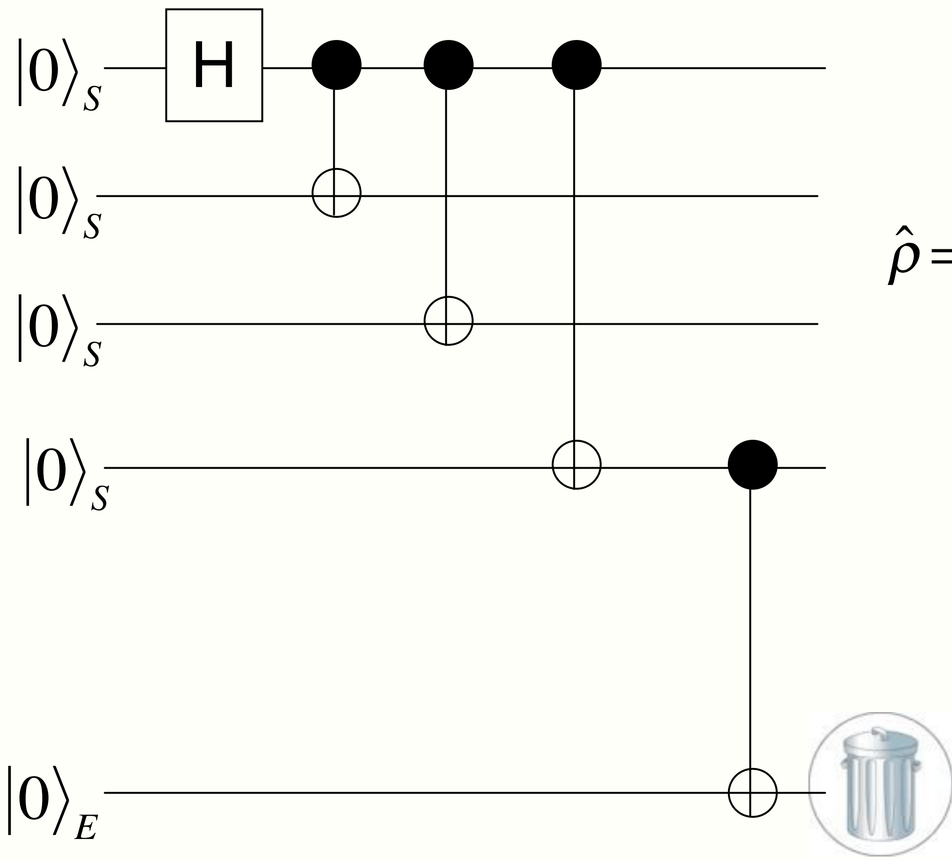
"Cat State" = N-qubit GHZ



$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0000\rangle + |1111\rangle\right)$$

Where are the Schrödinger Cats?

Totally mixed state



$$\hat{\rho} = \frac{1}{2}|0000\rangle\langle 0000| + \frac{1}{2}|1111\rangle\langle 1111|$$

Just one qubit interacting with the environment can decohere the whole state.

Implication for Quantum Computing Errors!

• Quantum algorithms rely on ***quantum parallelism***.

• Decoherence destroys interference between computational paths.

• The rate of decoherence can occur faster with the number of qubits (environment can distinguish a dead from a live cat much faster than a spin up vs. down nucleus).

Quantum Computing: Dream or Nightmare?

Classical Error Correction

- **Digital vs. Analog: robustness to noise!**
  - Bits stable to perturbations up to a threshold.

Error on a bit: Bit flip $0 \Rightarrow 1$, $1 \Rightarrow 0$

- **Protect against errors through redundancy.**

$$0_L \equiv 000,\ \ 1_L \equiv 111$$

With small probability *p* one bit flips

$$0_L \Rightarrow 001, 010, 100; \quad 1_L \Rightarrow 110, 101, 011$$

0_L and 1_L are still ***distinguishable***.

- Majority voting: Two out of three determine the logic state.
- Diagnose the error (minority) and recover (flip the bad egg).
- Code can correct for *single* bit-flip as long as $p<1/2$.

Quantum Error Correction

- **Digital vs. Analog: Which is it for quantum systems?**

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

continuous variables

- Continuous set of errors to correct?

- **No cloning theorem.**

$$|\psi\rangle \Rightarrow |\psi\rangle|\psi\rangle|\psi\rangle$$

- **Collapse of the wave function:**
  Measurement of a quantum bit can destroy the quantum coherence .

Quantum Error Correction

- **Digital vs. Analog: Which is it for quantum systems?**

It's a floor wax **AND** a dessert topping!!

As quantum are both particles and waves, quantum information is both analog AND digital.

- Analog.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

continuous variables

- Digital

If measured in standard basis, $|0\rangle \; or \; |1\rangle$

Errors can be ***discretized***!

No Cloning Theorem

Seek the following transformation:

$$|\psi\rangle|0\rangle \Rightarrow |\psi\rangle|\psi\rangle$$

fiducial state

All dynamical processes are ***linear*** *maps:*

Consider $|\psi\rangle = \alpha|\psi_1\rangle + \beta|\psi_2\rangle$

$$|\psi\rangle|0\rangle = \alpha|\psi_1\rangle|0\rangle + \beta|\psi_2\rangle|0\rangle \Rightarrow \alpha|\psi_1\rangle|\psi_1\rangle + \beta|\psi_2\rangle|\psi_2\rangle \neq |\psi\rangle|\psi\rangle$$

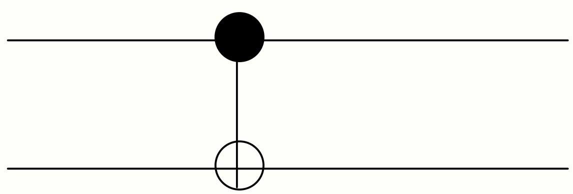
Quantum Copying

Distinguishable (orthogonal) states can be copied

$$|0\rangle|0\rangle \Rightarrow |0\rangle|0\rangle, \; |1\rangle|0\rangle \Rightarrow |1\rangle|1\rangle \qquad \text{CNOT}$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|0\rangle$$



$$\alpha|0\rangle|0\rangle + \beta|1\rangle|1\rangle$$

Quantum Three Qubit Bit-Flip Code

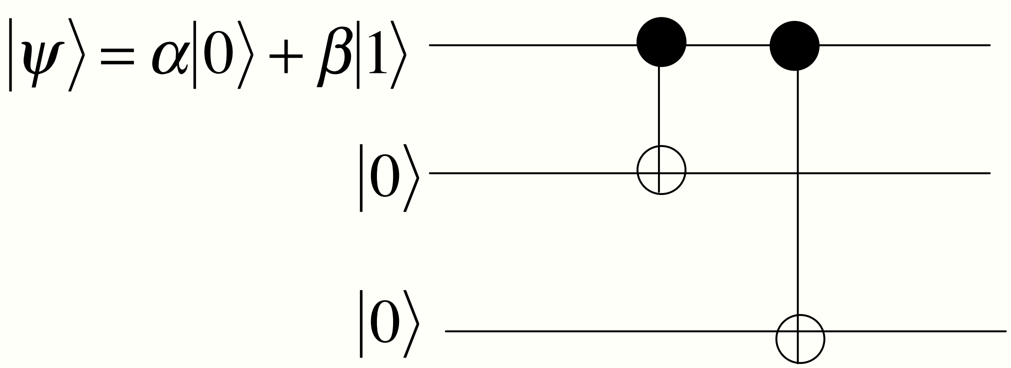
Possible Quantum Error:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \Rightarrow \alpha|1\rangle + \beta|0\rangle$$

Map qubit onto three qubit

Logical qubits: $|0\rangle_L = |0\rangle|0\rangle|0\rangle$

$$|1\rangle_L = |1\rangle|1\rangle|1\rangle$$



$$|\psi\rangle_{encoded} = \alpha|0\rangle_L + \beta|1\rangle_L$$

Measure the Error not the Data

We cannot measure whether a given physical qubit is $|0\rangle \; or \; |1\rangle$ without destroying the state.

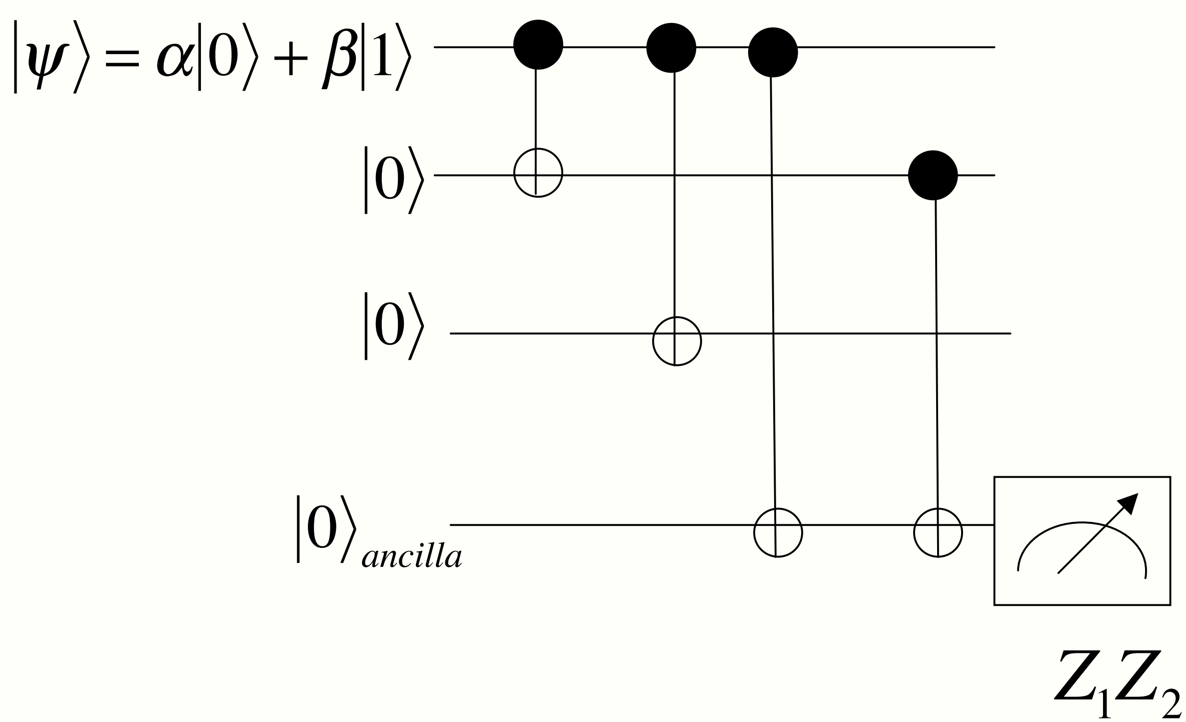
Measure a *joint property*:

Parity $Z_i Z_j$ +1 if bit i,j equal -1 unequal

Error diagnosis: Measure $Z_1 Z_2$ and $Z_2 Z_3$ (commuting)

Syndrome:	Recovery:
+1, +1	Do nothing
+1, -1	Flip bit 3
-1, +1	Flip bit 1
-1, -1	Flip bit 2

Performing a joint measurement



Quantum Three Qubit Phase-Flip Code

Possible Quantum Error:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \Rightarrow \alpha|0\rangle - \beta|1\rangle$$

Equivalent to bit flip in X-basis $|+\rangle \Rightarrow |-\rangle$, $|-\rangle \Rightarrow |+\rangle$

Map qubit onto three qubit

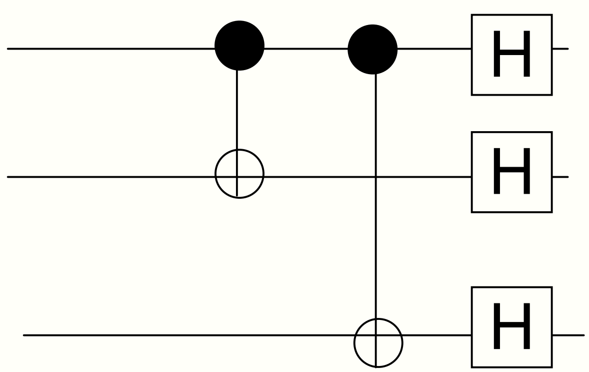
Logical qubits: $|0\rangle_L = |+\rangle|+\rangle|+\rangle$

$$|1\rangle_L = |-\rangle|-\rangle|-\rangle$$

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

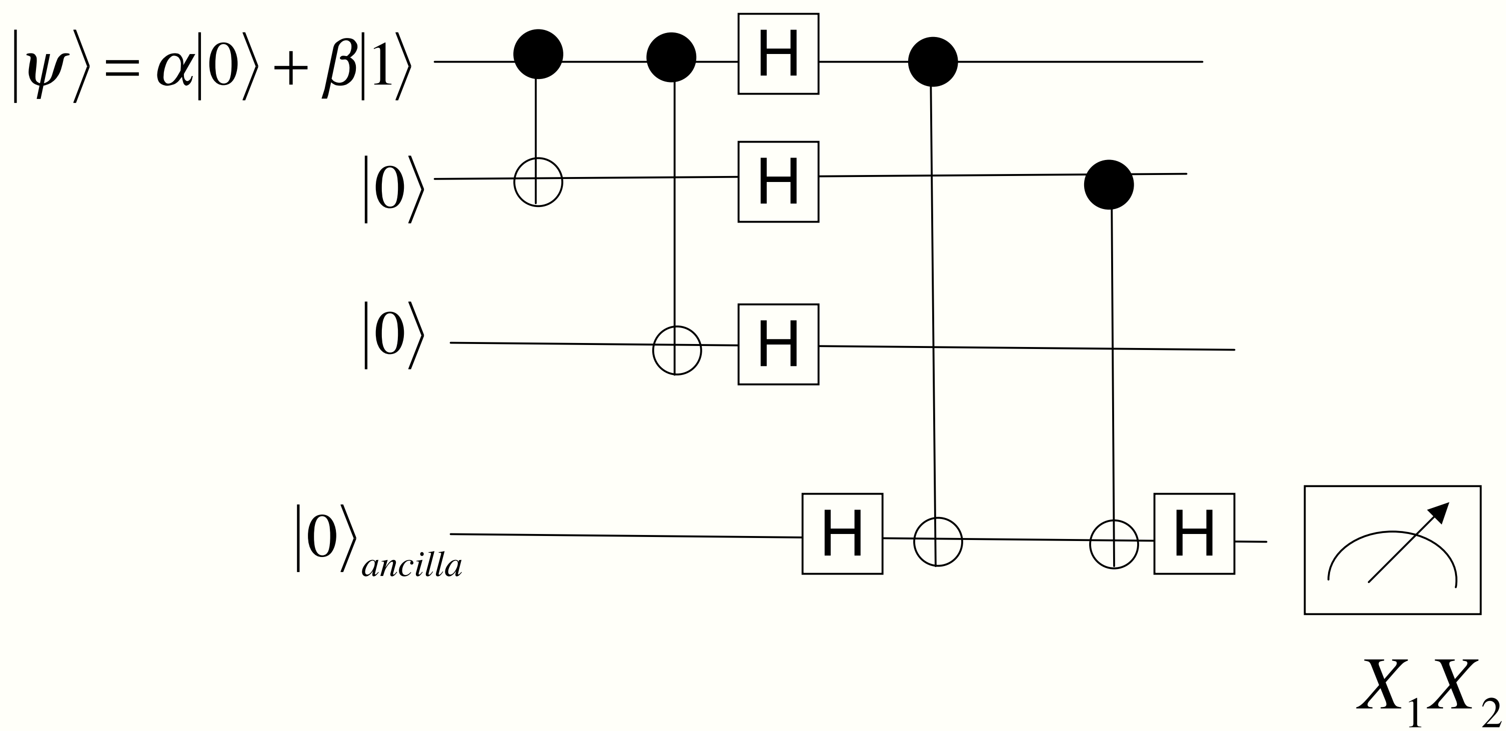
$|0\rangle$

$|0\rangle$



$$|\psi\rangle_{encoded} = \alpha|0\rangle_L + \beta|1\rangle_L$$

Syndrome Diagnosis



General signal qubit error

An *arbitrary single qubit error:*

Bit flip: X
Phase flip: Z
Both: Y = iXZ

Peter Shor: Lightning Strikes Twice

9 qubit error correcting code

$$|0\rangle_L = \left(|0\rangle|0\rangle|0\rangle + |1\rangle|1\rangle|1\rangle\right)\left(|0\rangle|0\rangle|0\rangle + |1\rangle|1\rangle|1\rangle\right)\left(|0\rangle|0\rangle|0\rangle + |1\rangle|1\rangle|1\rangle\right)$$

$$|1\rangle_L = \left(|0\rangle|0\rangle|0\rangle - |1\rangle|1\rangle|1\rangle\right)\left(|0\rangle|0\rangle|0\rangle - |1\rangle|1\rangle|1\rangle\right)\left(|0\rangle|0\rangle|0\rangle - |1\rangle|1\rangle|1\rangle\right)$$

General Error Correction

Basic Ingredients

- Define a "logical subspace".
- Discrete errors map to orthogonal subspaces.
- Measure the *subspace*, not the *state*.
- Subspace measurement = "Syndrome" diagnosis.
- Apply a recovery procedure conditional on syndrome.

Why does this work?

Logical states -- Entangled and nonlocal.
Errors -- Local and don't measure the state.

The Walmart approach: Encode globally, perturb locally.

Fault Tolerance

The procedure we described assume the syndrome diagnosis, and recovery were error free. To make the system fully "fault tolerant", we must:

- Account for errors in gate operations, measurement.
- Requires discrete set of gates.

Threshold theorem: If error probability per qubit is sufficiently small, can perform quantum computation forever. We can the threshold error probably $p_{threshold}$

The threshold rate depends heavily on the "error model".

Current thought -- for a depolarizing channel: $p_{threshold} > 10^{-5}$