

# Optimal classical-communication-assisted local model of $n$ -qubit Greenberger-Horne-Zeilinger correlations

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We present a model, motivated by the criterion of reality put forward by Einstein, Podolsky, and Rosen and supplemented by classical communication, which correctly reproduces the quantum-mechanical predictions for measurements of all products of Pauli operators on an  $n$ -qubit GHZ state (or “cat state”). The  $n - 2$  bits employed by our model are shown to be optimal for the allowed set of measurements, demonstrating that the required communication overhead scales linearly with  $n$ . We formulate a connection between the generation of the local values utilized by our model and the stabilizer formalism, which leads us to conjecture that a generalization of this method will shed light on the content of the Gottesman-Knill theorem.

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## I. INTRODUCTION

Bell’s theorem [1] codifies the observation that entangled quantum-mechanical systems exhibit stronger correlations than are achievable within any local hidden-variable (LHV) model. Beyond philosophical implications, the ability to operate outside the constraints imposed by local realism serves as a resource for information processing tasks such as communication [2], computation [3], and cryptography [4].

The violation of Bell-type inequalities demonstrates the in-principle failure of LHV models to account for all of the predictions of quantum mechanics. One approach to quantifying the observed difference between classically correlated systems and entangled states is to translate a quantum protocol involving entanglement into an equivalent protocol that utilizes only classical resources, e.g., the shared randomness of LHV’s and ordinary classical communication. Toner and Bacon [5] showed that the quantum correlations arising from local projective measurements on a maximally entangled state of two qubits can be simulated exactly using a LHV model augmented by just a single bit of classical communication. Pironio [6] took this analysis a step further, showing that the amount of violation of a Bell inequality imposes a lower bound on the average communication needed to reproduce the quantum-mechanical correlations.

The original Bell-type inequalities [1, 7] were formulated for pairs of qubits. Greenberger, Horne, and Zeilinger [8] introduced a qualitatively stronger test of local realism, based on a three-qubit state,  $|\psi_3\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ , which is now called the GHZ (or “cat”) state. Here  $|0\rangle(|1\rangle)$  represents the eigenvector of the Pauli  $Z$  op-

erator with eigenvalue  $+1(-1)$ . GHZ correlations have been experimentally demonstrated in entangled three-photon systems [9] and shown to be useful for performing information-theoretic tasks such as entanglement broadcasting [10] and quantum secret sharing [11].

Mermin [12] introduced a simple argument that shows how correlations between Pauli operators measured on a GHZ state violate local realism. We briefly review Mermin’s argument in Sec. II. Mermin’s formulation is based on the Einstein-Podolsky-Rosen (EPR) [13] reality criterion: “If, without in any way disturbing a system, we can predict with certainty ... the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.” This criterion is meant to capture what it means for a physical system to ‘possess’ a certain property.

Taking the EPR concept of an element of reality as our starting point, we formulate a LHV model for the  $n$ -qubit GHZ state. By itself, the model is inadequate. It cannot give the correct quantum-mechanical predictions for measurements of arbitrary products of Pauli operators and correlations among such measurements, as is clear from Mermin’s argument and its generalization to  $n$  qubits. Nonetheless, as we show in Sec. III, when the model is augmented by  $n - 2$  bits of classical communication, it does reproduce all the quantum-mechanical predictions for measurements of Pauli products and their correlations. We go on to prove in Sec. IV that this amount of classical communication is optimal for the allowed set of measurements, i.e., for measurements of Pauli products.

In Sec. V we demonstrate that our model arises naturally from a LHV simulation of a quantum circuit that creates the  $n$ -qubit GHZ state. The quantum circuit consists of an initial Hadamard gate  $H$  followed by a sequence of controlled-NOT (C-NOT) gates. It is a special case of a general class of quantum circuits identified by the Gottesman-Knill (GK) theorem [3]. The GK circuits are those composed of qubits (i) initially prepared

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in the state  $|00\dots 0\rangle$ , (ii) acted upon by gates in the Clifford group, which is generated by  $H$ ,  $90^\circ$  rotation about  $Z$ , and C-NOT [14], and (iii) subjected to measurements of products of Pauli operators. These circuits are capable of generating globally entangled states, such as the GHZ state, but their evolution can nevertheless be simulated in  $O(n^2/\log n)$  operations on a classical computer [3, 14, 15]. We return to the question of GK simulations in Sec. VI, comparing and contrasting them with our simulation of the creation of a GHZ state and speculating on how our results might impact understanding of the GK theorem.

## II. GHZ CORRELATIONS

Mermin's three-qubit GHZ argument can be summarized as follows. The three-qubit GHZ state  $|\psi_3\rangle$  is uniquely specified as the simultaneous  $+1$  eigenstate of a complete set of commuting Pauli products, one choice for which is  $\langle -XYY, -YXY, -YYX \rangle$ , where the ordering in the product specifies which qubit the Pauli operator applies to. In the language of the stabilizer formalism [3, 14], the three commuting operators are referred to as *stabilizer generators* of  $|\psi_3\rangle$ . The stabilizer generators give the definite outcome  $+1$  when measured, implying that a measurement of two of the Pauli operators in a generator can be used to predict the result of a measurement of the third with certainty. Thus, according to the EPR reality criterion, we should associate a local element of reality, having value  $+1$  or  $-1$ , with the  $X$  and  $Y$  Pauli operators of each qubit. Letting  $x_j$  and  $y_j$  denote the values of these six elements of reality, where  $j$  labels the qubit, the stabilizer generators require that  $x_1y_2y_3 = y_1x_2y_3 = y_1y_2x_3 = -1$ . Multiplying these three quantities together gives  $x_1x_2x_3 = -1$ , showing that the model predicts the result  $-1$  with certainty for a measurement of  $XXX$ . Because of the anticommutativity of the Pauli operators, however, the product of the stabilizer generators is  $+XXX$ , showing that quantum mechanics predicts the result  $+1$  for this measurement with certainty. Mermin's GHZ argument demonstrates the incompatibility of quantum theory with local realism.

The  $n$ -qubit GHZ state,  $|\psi_n\rangle = (|00\dots 0\rangle + |11\dots 1\rangle)/\sqrt{2}$ , is specified by  $n$  stabilizer generators  $\langle X^{\otimes n}, ZZI^{\otimes(n-2)}, ZIZI^{\otimes(n-3)}, \dots, ZI^{\otimes(n-2)}Z \rangle$ , where  $I$  is the identity operator. The full *stabilizer group* [3], generated by these generators, consists of the  $2^n$  commuting Pauli products of which  $|\psi_n\rangle$  is a  $+1$  eigenstate; it contains Pauli products that have (i) only  $I$ 's and an even number of  $Z$ 's and (ii) only  $X$ 's and an even number of  $Y$ 's, with an overall minus sign if the number of  $Y$ 's is not a multiple of 4. Of the  $2 \times 4^n$  Pauli products (including a  $\pm$  in front of the product),  $2^n$  are members of the stabilizer group,  $2^n$  are negatives of the stabilizer-group elements and thus yield  $-1$  with certainty when measured, and all the rest return  $\pm 1$  with equal proba-

	$X$	$Y$	$Z$
qubit 1	$R_2R_3 \cdots R_n$	$iR_1R_2 \cdots R_n$	$R_1$
qubit 2	$R_2$	$iR_1R_2$	$R_1$
qubit 3	$R_3$	$iR_1R_3$	$R_1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
qubit $n$	$R_n$	$iR_1R_n$	$R_1$

TABLE I: Table of LHV's associated with an  $n$ -qubit GHZ state. Each row corresponds to a qubit, and each column to a measurement. The quantities  $R_j$  denote classical random variables that return  $\pm 1$  with equal probability. The origin and meaning of the subscripts  $j$  becomes clear when we consider the creation of a GHZ state in Sec. V. The outcome predicted for a joint measurement of a Pauli product is obtained by multiplying the corresponding table entries for each qubit (using 1 for unmeasured qubits, i.e., for an identity operator appearing in the Pauli product) and discarding any factor of  $i$  in the final product. For example, for a joint measurement of  $XYX$  on the  $(n=3)$ -qubit GHZ state, our model predicts the outcome  $(R_2R_3)(iR_1R_2)(iR_1R_3) = -1$ , in agreement with quantum mechanics. Here we have used the fact that  $R_j^2 = 1$ . Similarly, for a measurement of  $IYZ$ , the product of table entries is  $(iR_1R_2)(R_1) = iR_2$ ; with the  $i$  discarded, the predicted outcome is the random result  $R_2$ , again in accord with quantum mechanics. The use of  $i$  does not mean that the results of  $Y$  measurements are imaginary; rather the  $i$  is a "flag" that tells us how to combine  $Y$  values for different qubits in a joint measurement. Although the LHV table might seem not to respect the qubit-exchange symmetry of the GHZ state, one easily sees that it does by defining  $R'_2 = R_2 \cdots R_n$ , which exchanges the roles of the first and second qubits in the table.

bility.

Mermin's argument generalizes straightforwardly to  $|\psi_n\rangle$  (our proof of optimality in Sec. IV can be viewed as just such a generalization) and shows that no local realistic model can correctly predict the outcomes of all measurements of products of Pauli operators performed on  $|\psi_n\rangle$  and correlations among such measurements. We now present a classical-communication-assisted LHV model that does yield all of the correct quantum-mechanical predictions.

## III. COMMUNICATION-ASSISTED LOCAL MODEL OF GHZ CORRELATIONS

Our LHV model is specified in Table I, which lists local realistic values for the  $X$ ,  $Y$ , and  $Z$  Pauli operators of each qubit. The caption describes how to determine the predicted outcome for a measurement of any Pauli product by multiplying the appropriate table entries and discarding any factor of  $i$  in the final product. The use of the imaginary phase  $i$  in the  $Y$  column, apparently just a curiosity, actually plays a crucial role. It reconciles some of the conflicting predictions of commuting LHV's and anticommuting Pauli operators, which form the basis of Mermin's GHZ argument. More precisely, the multi-

plicative algebra of these phases provides a concise representation of the  $n - 2$  bits of classical communication required to ensure that our LHV model yields all of the correct quantum-mechanical predictions.

To show that Table I gives correct predictions for measurements of Pauli products, we consider those measurements for which the table predicts a definite outcome. Suppose first that a Pauli product contains no  $X$ 's or  $Y$ 's, but consists solely of  $I$ 's and  $Z$ 's. Then it is clear that the table predicts certainty, with the outcome being  $+1$ , if and only if the number of  $Z$ 's in the product is even. Suppose now that the product has an  $X$  or a  $Y$  in the first position. Then it is apparent that to avoid a random variable in the overall product, all of the other elements in the product must be  $X$ 's or  $Y$ 's and the number of  $Y$ 's must be even; the outcome is  $+1$  if the number of  $Y$ 's is a multiple of 4 and  $-1$  otherwise. Finally, suppose the Pauli product has an  $X$  or a  $Y$  in a position other than the first. Then the only way to avoid a random variable in the overall product is to have an  $X$  or a  $Y$  in the first position, and we proceed as before. These considerations show that our model predicts a definite outcome, with the correct sign, for precisely those Pauli products that are in the stabilizer group (or their negatives), including as a special case the observables forming the basis of Mermin's GHZ argument. Likewise, the model correctly predicts a random result for all other Pauli products.

Our model correctly predicts the outcomes for all measurements of Pauli products, including single-qubit measurements. It fails, however, in some of its predictions for correlations between single-qubit measurements. To be correct, the model would have to reproduce all these correlations for all sets of single-qubit measurements. The model fails because products of single-qubit measurement results predicted by the model are not always equal to the corresponding joint measurement results. This inconsistency is a direct consequence of the rule that discards  $i$  from a calculated measurement outcome. As an example, consider the single-qubit measurements  $XII$ ,  $IYI$ , and  $IYY$  on a three-qubit GHZ state. The product of the single-qubit measurement results,  $R_2R_3$  for  $XII$ ,  $R_1R_2$  for  $IYI$ , and  $R_1R_3$  for  $IYY$  is  $+1$ , which is inconsistent with the prediction of the model and of quantum mechanics for a joint measurement of the observable  $XYI$ . From the perspective of the model, classical communication between qubits, an obviously nonlocal element, is necessary precisely to ensure the consistency of joint measurement predictions with products of single-qubit predictions.

The inconsistency between joint and correlated local predictions is a general feature of our LHV model for  $n$ -qubit GHZ states. It occurs only for joint measurements that involve  $Y$  measurements on some qubits and that have a definite outcome, i.e., joint measurements of stabilizer elements that contain  $Y$ 's. Joint measurements that yield a random result do not suffer from this problem because the randomness of a product is unaffected by discarding  $i$ 's. More precisely then, the inconsistency

occurs only for joint measurements that are products of  $X$ 's and  $Y$ 's on all the qubits, with the number  $Y$ 's being an even number that is not a multiple of 4.

The protocol for ensuring consistency between joint and composite local predictions proceeds as follows. An observer called Alice, stationed at, say, the first qubit, is put in charge of ensuring consistency with single-qubit measurements. She does so by changing or not changing the sign of the outcome on her qubit, based on what is measured on her qubit and information she receives about what is measured on the other qubits. Because of the qubit-exchange symmetry of the GHZ state (and of the LHV table), an observer stationed at any qubit could play the role of Alice. Alice ensures consistency by changing the sign of her local outcome if and only if (i) a measurement of  $X$  or  $Y$  is made on her qubit and (ii) the total number of  $Y$  measurements on all qubits is an even number that is not a multiple of 4. The protocol requires  $n - 1$  bits of communication as each of the other qubits reports to Alice whether  $Y$  was measured on that qubit. The protocol clearly fixes all those cases that need correction; just as important, in all situations where Alice flips her qubit, all subsets of qubits that include Alice's qubit, except for the case of a needed correction, have a random measurement product, which is therefore unaffected by Alice's flip. The success of this protocol clearly relies on very special properties of the stabilizer group for the  $n$ -qubit GHZ state.

We can put the protocol in a more mathematical form by letting  $r_1 = 1$  if an  $X$  or  $Y$  measurement is made on the first qubit and  $r_1 = 0$  otherwise and by letting  $q_j = i$  if  $Y$  is measured on the  $j$ th qubit and  $q_j = 1$  otherwise. Alice ensures consistency by flipping her local outcome if and only if  $p_n = r_1q_1 \cdots q_n = -1$ . This formulation allows us to see easily that we can do a bit better than the  $n - 1$  bits required by the original protocol. The key is to notice that when  $p_n = \pm i$ , all subsets of qubits that include Alice's qubit have a random measurement product, so a flip by Alice goes unnoticed. As a result, Alice can get by with the truncated product  $p_{n-1} = r_1q_1 \cdots q_{n-1}$ , flipping her local outcome if and only if  $p_{n-1} = i$  or  $-1$ . This scheme requires the promised  $n - 2$  bits of communication, because Alice doesn't need to know whether a  $Y$  measurement is made on the  $n$ th qubit; it works because Alice flips whenever  $p_n = -1$ , as required, with the additional flips when  $p_n = \pm i$  not doing any harm.

The consistency scheme generalizes trivially to the case of Pauli-product measurements made on  $l$  disjoint sets of qubits. For each set  $k$  chosen from the  $l$  sets, the table yields a measurement product that is the predicted outcome multiplied by  $q_k = i$  or  $q_k = 1$ . Putting Alice in charge of the first set, all but the last of the other sets communicates  $q_k$  to Alice, who computes the product  $r_1q_1 \cdots q_{l-1}$ , where  $r_1 = 0$  if no measurement or a  $Z$  measurement is made on any qubit in her set and  $r_1 = 1$  otherwise. Alice flips her set's outcome if and only if  $r_1q_1 \cdots q_{l-1} = i$  or  $-1$ . Consistency is thus ensured at the price of  $l - 2$  bits of communication.

#### IV. PROOF OF OPTIMALITY

Using an elaboration of Mermin's GHZ argument [12], we now demonstrate that our model is optimal by showing that any protocol that is allowed at most  $n - 3$  bits of classical communication is incapable of yielding all quantum-mechanical predictions for measurements of Pauli products on  $|\psi_n\rangle$  and their correlations. For this purpose, imagine the  $n$  qubits as the nodes of a graph; two qubits are connected by a line if at least one bit is communicated between them. The graph partitions the qubits into disjoint connected subsets. There being at most  $n - 3$  lines, it follows that there are at least three disconnected subsets, since at best each line consolidates two subsets into one, thereby eliminating one subset. Moreover, it is always possible to arrange the communication so that there are three subsets. The communication can do no more than allow us to treat all Pauli products within a subset as a single joint observable. We can restrict attention to the case of three subsets, since amalgamating disconnected subsets allows the communication more power than it actually has.

The situation then is that we have three disconnected subsets, containing  $k$ ,  $l$ , and  $m$  qubits, with  $k+l+m = n$ . Since the GHZ state is invariant under qubit exchange, we can make the first  $k$  qubits those in the first subset and the next  $l$  qubits those in the second subset, leaving the final  $m$  qubits to be those in the third subset. We now define six Pauli products:  $A = X^{\otimes(k-1)}X$  and  $B = X^{\otimes(k-1)}Y$  for the first subset,  $C = X^{\otimes(l-1)}X$  and  $D = X^{\otimes(l-1)}Y$  for the second subset, and  $E = X^{\otimes(m-1)}X$  and  $F = X^{\otimes(m-1)}Y$  for the third subset. The four operators,  $ACE$ ,  $-ADF$ ,  $-BCF$ ,  $-BDE$ , are in the stabilizer group of  $|\psi_n\rangle$  and thus give a definite outcome  $+1$  when measured, implying that a measurement of any two of the operators in the product can be used to predict with certainty the result of a measure of the third. The EPR reality criterion then says that we should associate elements of reality, having values  $\pm 1$ , with the six Pauli products  $A-F$ . Denoting the values of these elements of reality in the obvious way, the definite values of the last three stabilizer elements imply that  $adf = bcf = bde = -1$ . The product of these three quantities is  $ace = -1$ , contradicting the  $+1$  prediction of quantum mechanics for a measurement of  $ACE$ .

For completeness, we note another form of the argument. According to the elements of reality, the observable  $\mathcal{M} = ACE - ADF - BCF - BDE$  has the value

$$\mathcal{M} = ace - adf - bcf - bde = c(ae - bf) - d(af + be). \quad (1)$$

Since  $ae = \pm bf \iff af = \pm be$ , it is easy to see that  $\mathcal{M} = \pm 2$ . This implies that the expectation value satisfies  $|\langle \mathcal{M} \rangle| \leq 2$ , whereas the  $n$ -qubit GHZ state has  $\langle \mathcal{M} \rangle = 4$ . This form of the argument does not make use of the properties of the GHZ state, and it makes clear that stochastic models can do no better than the deterministic models considered here. We also note that

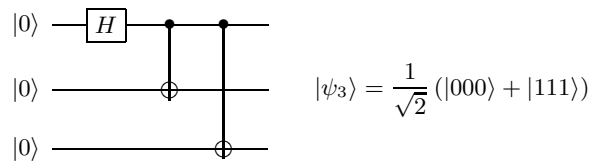


FIG. 1: Quantum circuit that generates the three-qubit GHZ state.

this argument produces a Bell inequality with auxiliary communication[16].

#### V. QUANTUM CIRCUIT

Table I is the basis of our communication-assisted LHV model. It arises naturally from a quantum circuit that creates the  $n$ -qubit GHZ state from an initial state  $|00\dots 0\rangle$ . One such circuit consists of a Hadamard gate on the first qubit followed by  $n - 1$  C-NOT gates, with the leading qubit being the control and the remaining qubits serving successively as targets. The three-qubit version of this circuit is shown in Fig. 1. The Hadamard gate  $H$  transforms the Pauli operators according to

$$HXH^\dagger = Z, \quad HYH^\dagger = -Y, \quad HZH^\dagger = X. \quad (2)$$

Similarly, under the action of C-NOT, we have

$$\begin{aligned} C(XI)C^\dagger &= XX, & C(YI)C^\dagger &= YX, & C(ZI)C^\dagger &= ZI, \\ C(IX)C^\dagger &= IX, & C(IY)C^\dagger &= ZY, & C(IZ)C^\dagger &= ZZ, \end{aligned} \quad (3)$$

where the first qubit is the control and the second is the target. These operator transformations lead to the table update rules given in Fig. 2, which traces the evolution of the LHV table during the creation of a three-qubit GHZ state using the circuit of Fig. 1. A simple generalization to  $n$  qubits leads to Table I for the  $n$ -qubit GHZ state.

The C-NOT update rules given in Fig. 2 must be consistent with the fifteen nontrivial transformations of Pauli products generated by  $C$ . Six of these transformations, listed in Eq. (3), serve as the basis for the update rules. Because  $C = C^\dagger$ , the rules are automatically consistent with four other transformations. In addition, the rules are clearly consistent with the transformation  $C(ZX)C^\dagger = ZX$ . Consistency with the remaining four transformations,  $C(XY)C^\dagger = YZ$ ,  $C(XZ)C^\dagger = -YY$ , and their inverses, requires that

$$\begin{aligned} X_c^I Y_t^I &= Y_c^F Z_t^F = Y_c^I Z_c^I Z_t^I X_t^I, \\ X_c^I Z_t^I &= -Y_c^F Y_t^F = -Y_c^I Z_c^I X_t^I Y_t^I. \end{aligned} \quad (4)$$

These relations do not hold generally, but are satisfied if the initial entries for both the control and target are correlated according to  $XYZ = i$  (or  $XYZ = -i$ ), with  $X$  and  $Z$  real and  $Y$  imaginary. These conditions hold in all our applications of C-NOT. It is for this reason that

$$\begin{array}{ccc}
\begin{array}{ccc} X & Y & Z \\ R_1 & -iR_1 & 1 \\ R_2 & iR_2 & 1 \\ R_3 & iR_3 & 1 \end{array} & \xrightarrow{H_1} & \begin{array}{ccc} X & Y & Z \\ 1 & iR_1 & R_1 \\ R_2 & iR_2 & 1 \\ R_3 & iR_3 & 1 \end{array} \\
& & \xrightarrow{\text{C-NOT}_{12}} & \begin{array}{ccc} X & Y & Z \\ R_2 & iR_1R_2 & R_1 \\ R_2 & iR_1R_2 & R_1 \\ R_3 & iR_3 & 1 \end{array} \\
& & & \xrightarrow{\text{C-NOT}_{13}} & \begin{array}{ccc} X & Y & Z \\ R_2R_3 & iR_1R_2R_3 & R_1 \\ R_2 & iR_1R_2 & R_1 \\ R_3 & iR_1R_3 & R_1 \end{array}
\end{array}$$

FIG. 2: Evolution of the LHV table during the creation of a three-qubit GHZ state using the circuit of Fig. 1. The initial table yields the correct quantum predictions for the state  $|000\rangle$ . The rules for updating the table through Hadamard and C-NOT gates come from the operator transformations (2) and (3). The Hadamard update rules are

$$X^F = Z^I, \quad Y^F = -Y^I, \quad Z^F = X^I,$$

where I and F denote the initial and final values of a table entry, before and after the application of the gate. The rules for updating through a C-NOT, with control  $c$  and target  $t$ , are

$$\begin{aligned}
X_c^F &= X_c^I X_t^I, & Y_c^F &= Y_c^I X_t^I, & Z_c^F &= Z_c^I, \\
X_t^F &= X_t^I, & Y_t^F &= Z_c^I Y_t^I, & Z_t^F &= Z_c^I Z_t^I.
\end{aligned}$$

The update rules are local in that they only require changes to table entries corresponding to the qubits involved in a gate. The subscripts on the random variables in Table I are now seen to represent the qubits to which these variables were initially associated. Correlations arising from the application of C-NOT gates then correspond to pairs of identical subscripts.

the initial sign of the  $Y$  entry for the first qubit (see the first table in Fig. 2) is opposite that of the remaining qubits.

## VI. CONCLUSION

We have shown that it is possible to reproduce correctly the quantum-mechanical measurement predictions for the set of all  $n$ -fold products of Pauli operators on an  $n$ -qubit GHZ state using only Mermin-type LHV's and  $n - 2$  bits of classical communication. The  $n - 2$  bits of communication, shown here to be optimal, are required to ensure that the products of local measurement predictions are consistent with the corresponding joint predictions. We also show how our model arises naturally

from a simulation of a quantum circuit that creates the  $n$ -qubit GHZ state.

The circuit that creates the GHZ state is an example of the Gottesman-Knill circuits mentioned in the Introduction. Although GK circuits can produce global entanglement, as in the  $n$ -qubit GHZ state, they can be efficiently simulated in  $O(n^2/\log n)$  steps on a classical computer [3, 14, 15]. The important difference between the GK simulation of the circuit that creates the  $n$ -qubit GHZ state and our communication-assisted LHV simulation is that the GK algorithm tracks the evolution of *nonlocal* hidden variables, represented by the  $n$  generators of the stabilizer group, and therefore requires no communication overhead to give correct predictions.

A generalization of our results might lead to a new perspective on the GK theorem since our results imply that, at least in this limited case, we can replace the nonlocal hidden variables represented by the stabilizer generators with LHV's and a linear amount of classical communication. We conjecture that this is a generic feature of quantum circuits obeying the constraints of the GK theorem; that is, we expect that any quantum state produced by a GK circuit can be modelled with EPR elements of reality plus an amount of classical communication that scales linearly in the number of qubits.

There are two main obstacles to a straightforward extension of our LHV model to general GK circuits. The first is the difficulty, expressed in Eq. (4), in maintaining the consistency conditions for the C-NOT update rules. The second is the reliance of our communication protocol on very special properties of the GHZ state; for general GK states, the communication protocol will have to be more complicated, with a proof of optimality correspondingly more complicated as well. Nonetheless, the existence of a communication-assisted LHV model for arbitrary GK circuits and the entangled states they produce is currently under investigation.

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