

Continuous measurement with traveling-wave probes

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We consider the use of a traveling-wave probe to continuously measure the quantum state of an atom in free space. Unlike the more familiar cavity QED geometry, the traveling wave is intrinsically a multimode problem. Using an appropriate modal decomposition, we determine the effective measurement strength for different atom-field interactions and different initial states of the field. These include the interaction of a coherent-state pulse with an atom, the interaction of a Fock-state pulse with an atom, and the use of Faraday rotation of a polarized laser probe to perform a quantum nondemolition measurement on an atomic spin.

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I. INTRODUCTION

Current research to create quantum information processors has forced a reexamination of the underlying description of these devices. In order for us to gain information about quantum systems they must be measured. Whereas the standard picture of such measurements involves the “collapse of the wave function” following Von Neumann’s projection postulate, such strongly measuring probes are rarely implemented in the laboratory. More typically, a continuous probe interacts with the system which is then detected as a macroscopic signal. Examples include the probing of a quantum dot with a single electron transistor [1] and the measurement of the position of a micromechanical cantilever by monitoring the modulation of a reflected laser beam [2]. The formalism of quantum mechanics provides a number of different approaches for analyzing such situations. Scattering theory employs Green’s function input-output relations to describe the evolution of the probe asymptotically, both before and after its interaction with the system under examination. Alternatively, the theory of quantum trajectories [3] provides a dynamical description of the quantum system being measured, conditioned on the continuous information being collected via the probe. As laboratory developments give us access to the control and manipulation of quantum systems, these descriptions become ever more relevant. The quantum trajectory approach has the advantage of directly tying the dynamics of the system’s evolution to the measurement record. The ability to do this is essential when implementing adaptive measurement and control strategies employing feedback [4–7].

The standard paradigm for continuous measurement is cavity QED. The dynamics of a cavity mode of the electromagnetic field (perhaps coupled to an atom) are monitored by a partially transmitting mirror [8,9]. Input-output scattering theory, suited specifically to the language of optical elements [10], is used to connect the intracavity dynamics with those of the traveling signal. In order to translate the typically discrete information of the individually transmitted photons into continuous information one considers a homodyne or heterodyne measurement, in which the signal is

mixed with a macroscopic local oscillator. The result is a stochastic Schrödinger equation which describes both the localization of the quantum state conditioned on the measurement and also the effect of quantum “back-action noise” [11].

In many applications, there is no confining cavity to set the interaction strength between the system under observation and the probe. Though a “quantization volume” may be employed as a calculational tool, its imagined characteristics should not determine physically relevant quantities. For example, Milburn *et al.* [2] modeled continuous observation of a moving cantilever monitored by the modulation of a reflected laser beam. They considered the cantilever to be a mirror, which, partnered with an imagined partially transmitting surface, formed a leaky optical cavity. Under the assumption that the transmission rate of light through the fictitious mirror is much faster than the characteristic rate at which the cantilever moves, the cavity could be adiabatically eliminated from the dynamics. This led to a stochastic Schrödinger equation for the continuously observed cantilever alone. The unphysical quantization volume, however, still appeared implicitly in this equation.

The key parameter that characterizes the dynamics of a continuously observed system is the “measurement strength,” which we will denote by κ . It determines the rate at which information is gathered about the system and consequently, sets the scale at which effects such as quantum back action become significant. For example, Bhattacharya *et al.* [12] have shown that for sufficiently macroscopic systems there is a window of values for κ such that continuous observation can localize the probability distribution to a quantum trajectory that faithfully tracks the classically predicted trajectory, with minimal quantum noise. Another example is the continuous measurement of ensembles of atoms, controlled through their collective interaction with a common probe, to produce nonclassical spin-squeezed states [13,14]. These effects depend crucially on κ and its relation to the other rates governing the system dynamics.

In this paper, we formulate the problem of continuous measurement by a traveling-wave probe. We derive a master equation describing the situation in which the system is monitored by the probe, but the measurement result is not recorded. This allows us to identify the important characteristic scales of the problem without reference to a particular

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measurement scheme. We begin in Sec. II by establishing the necessary formalism for treating propagating fields in quantum optics, in contrast to the more familiar closed cavity problems. We apply this formalism in Sec. III to the fundamental problem of a two-level atom interacting with a resonant laser field. When the field is treated classically this leads to Rabi oscillations, but when treated quantum mechanically the laser not only manipulates the atom but also acts to continuously measure it. We determine the rate at which the measurement back action leads to decoherence in the atomic system. This is contrasted with the evolution when the atom is coupled to a resonant electromagnetic pulse with a fixed photon number n . In particular, we explore the circumstances under which we can recover the usual Jaynes-Cummings solution for a two-level atom coupled to a single mode [15] and show how the behavior for a single photon diverges from this solution. In Sec. IV, we consider continuous measurement of an atomic spin through the Faraday rotation of an off-resonant laser field. This process has been used to generate spin-squeezed states in atomic ensembles [13,14]. We conclude and summarize our results in Sec. V.

II. QUANTUM DESCRIPTION OF PROPAGATING FIELDS

Classically, when considering quasimonochromatic propagating fields, it is natural to model the evolution of the system as a function of the propagation direction z . The field at z can then be decomposed into a complete set of orthonormal temporal modes which act locally. One might be tempted to describe the quantum fields in an analogous manner by quantizing the temporal modes at fixed position, $[\hat{a}(t), \hat{a}^\dagger(t')] = \delta(t-t')$. The field operator could then be decomposed into a complete set of orthonormal mode functions $\phi_i(t)$, so that $\hat{a}(t) = \sum_i \phi_i(t) \hat{c}_i$, with $[\hat{c}_i, \hat{c}_j^\dagger] = \delta_{ij}$. Boundary conditions at some initial plane could then be used to restrict the mode content, possibly to a single temporal mode.

This approach was taken by van Enk and Kimble [16] and also by Gea-Banacloche [17] who considered an analogous problem to the one we address here. They studied how errors were generated in quantum logic operations due to the fact that control pulses are not truly classical and can become entangled with the atoms with which they are interacting. Their analysis led to an effective single temporal mode theory. Though some of their conclusions are correct, one must take great care to understand the regimes under which this formalism is applicable. Consider, for example, a single-photon pulse interacting with a localized two-level atom. Let us suppose that the duration of the pulse is short compared to the natural lifetime of the atom in its excited state but sufficiently long to be considered quasimonochromatic and on resonance. Defining creation and annihilation operators for the single temporal mode associated with this pulse, the Hamiltonian, in the rotating wave approximation, appears to have the familiar Jaynes-Cummings form

$$\hat{H} = \hbar g (\hat{a} \hat{\sigma}_+ + \hat{a}^\dagger \hat{\sigma}_-), \quad (1)$$

where $\hat{\sigma}_\pm$ are the usual raising and lowering operators asso-

ciated with the two-state atom. Given the atom initially in its ground state, the solution leads to quantum Rabi oscillations,

$$|\psi(t)\rangle = \cos(gt)|g\rangle|1\rangle - i \sin(gt)|e\rangle|0\rangle. \quad (2)$$

This falsely predicts the possibility of a single-photon 2π pulse in free space, whereby the photon is perfectly absorbed and then reemitted into the original mode. In reality once the atom has absorbed the photon it will reemit into a mode consistent with its radiation pattern, *not* into the initial packet mode. That is, the single photon will be *scattered*. This emission must also obey causality; no information about the emitted photon can register on a distant detector at a spacelike separated point. In the solution above, however, the atom both absorbs from and emits into a spatially delocalized photon mode in free space, violating causality.

The problems with causality arise from the faulty quantization procedure outlined above. Quantum fields must be defined over all space at an initial time (more generally on an initial spacelike surface). Unitarity then ensures that *equal-time*, not *equal-space*, commutation relations are preserved. Nonequal-time commutation relations cannot generally satisfy the canonical commutation relations, being inconsistent with Poincaré invariance [18]. The exception is for free fields or fields that behave like them (e.g., fields traveling through matter whose response is approximately linear, or asymptotic “in” and “out” fields as used in scattering theory).

We review here a formalism appropriate for treating the quantum optics of paraxial propagating fields [19]. Consider a quasimonochromatic paraxial beam with frequency ω_0 and wave number k_0 . We write the positive frequency component

$$E^{(+)}(\mathbf{x}, t) = \mathcal{E}(z, t) \phi_T(x, y) e^{i(k_0 z - \omega_0 t)}, \quad (3)$$

where $\exp[i(k_0 z - \omega_0 t)]$ is the “carrier wave,” $\phi_T(x, y)$ is the “transverse mode” (e.g., Gaussian), and $\mathcal{E}(z, t)$ is the slowly varying envelope, meaning its spatiotemporal variation is much slower than the carrier wavelength or frequency. We have ignored both diffraction and the vector nature of the field. It is easy to show that the free space wave equation becomes

$$\left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial z} \right) \mathcal{E}(z, t) = 0, \quad (4)$$

for the envelope, whose solution is $\mathcal{E}(z, t) = \mathcal{E}(z - ct, 0)$, i.e., propagation of the pulse envelope. We quantize by replacing the field envelope with a scaled operator

$$\mathcal{E}(z, t) \Rightarrow \sqrt{2\pi\hbar\omega_0} \hat{\Psi}(z, t), \quad (5)$$

which satisfies the canonical equal-time commutation relation [19]

$$[\hat{\Psi}(z, t), \hat{\Psi}^\dagger(z', t)] = \delta(z - z'). \quad (6)$$

This commutator is equivalent to that of a nonrelativistic massive Bose gas in one dimension. Here, the carrier wave plays the role of the rest mass and the slowly varying enve-

lope plays the role of small fluctuations around the mass shell. The free field Hamiltonian (removing the carrier wave energy) takes the form

$$\hat{H}_F = c \int dz \hat{\Psi}^\dagger(z) \left(-i\hbar \frac{\partial}{\partial z} \right) \hat{\Psi}(z), \quad (7)$$

whose Heisenberg equation of motion gives the wave equation above. This Hamiltonian is nothing but the second-quantized version of the energy of a photon $E = cp$.

Consider an atom interacting with the field. In the electric-dipole and rotating wave approximation the interaction Hamiltonian is

$$\hat{H}_{AF} = \int d^3x |\Phi(\mathbf{x})|^2 d [\hat{E}^{(+)}(\mathbf{x}) \hat{\sigma}_+ + \hat{E}^{(-)}(\mathbf{x}) \hat{\sigma}_-], \quad (8)$$

where d is the dipole matrix element and $\Phi(\mathbf{x})$ is the atom's center-of-mass wave function. We take the atom to be trapped, having center-of-mass wave function $\Phi(\mathbf{x}) = f_T(x, y) f_L(z)$ [20]. Then, the interaction Hamiltonian becomes

$$\hat{H}_{AF} = d \sqrt{\frac{2\pi\hbar\omega_0}{A}} \int dz |f_L(z)|^2 [\hat{\Psi}(z) \hat{\sigma}_+ + \hat{\Psi}^\dagger(z) \hat{\sigma}_-],$$

where

$$\int dx dy |f_T(x, y)|^2 \phi_T(x, y) \equiv \frac{1}{\sqrt{A}}, \quad (9)$$

A being the effective area of the mode interacting with the atom. Let us go to the interaction picture by including the free evolution of the atom and field in the interaction Hamiltonian. Assuming the carrier wave is on resonance

$$\begin{aligned} \hat{H}_{AF}(t) = d \sqrt{\frac{2\pi\hbar\omega_0}{A}} \int dz |f_L(z)|^2 [\hat{\Psi}(z-ct) \hat{\sigma}_+ \\ + \hat{\Psi}^\dagger(z-ct) \hat{\sigma}_-]. \end{aligned} \quad (10)$$

Finally, given a set of orthonormal functions ("longitudinal modes") $\{\phi_i(z)\}$, chosen to be real without loss of generality,

$$\hat{\Psi}(z) = \sum_i \phi_i(z) \hat{a}_i, \quad (11)$$

$$\begin{aligned} \hat{H}_{AF}(t) = d \sqrt{\frac{2\pi\hbar\omega_0}{A}} \sum_i \int dz |f_L(z)|^2 \phi_i(z-ct) \\ \times (\hat{a}_i \hat{\sigma}_+ + \hat{a}_i^\dagger \hat{\sigma}_-). \end{aligned} \quad (12)$$

The Hamiltonian in Eq. (12) describes the interaction of each longitudinal mode as it propagates past the atom (Fig. 1). Assuming this time scale is much shorter than any other dynamical scale in the problem, it is appropriate to make the Markov approximation, whereby one coarse grains over the time scale within which the system and reservoir retain

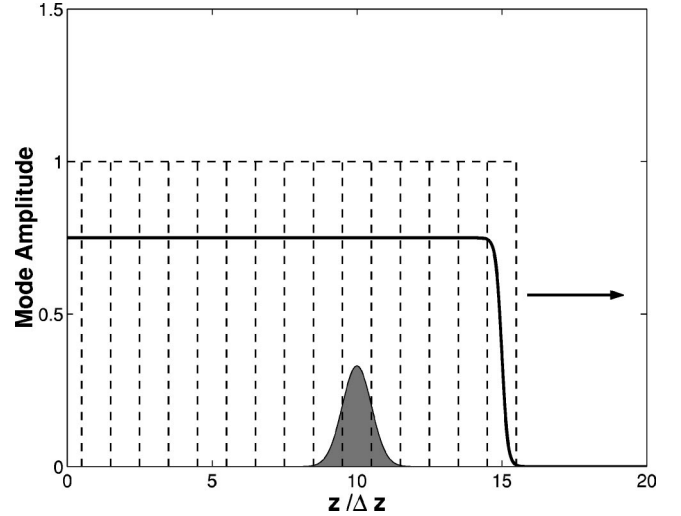


FIG. 1. An initial traveling-wave pulse (solid line) is broken up into many smaller modes $\{\phi_i(z)\}$ (dashed line). The wave packet of the interacting atom (gray) has the same width as the mode functions Δz .

"memory." To this end, we breakup the z axis into slices of size Δz , each the extent of the atom wave packet (e.g., the rms of the probability density). The Markov approximation will hold if the transit time of the field across the atomic packet, $\Delta t = \Delta z/c$, is much smaller than any time scale over which the atom changes. This is certainly an excellent approximation. Since we will not consider dynamics on a time scale smaller than Δt , we can approximate the set of atomic wave packets, centered at each coarse-grained slice, as a complete orthonormal set. That is, each slice is an approximate δ function. Then, normalization of both the atomic wave packet and mode functions combine to give

$$\int dz |f_L(z)|^2 \phi_i(z-ct) = \frac{1}{\sqrt{\Delta z}} \Theta_i(t), \quad (13a)$$

$$\Theta_i(t) = \begin{cases} 1, & (i-1)\Delta t < t \leq i\Delta t \\ 0, & \text{otherwise.} \end{cases} \quad (13b)$$

Under this approximation, the Hamiltonian takes the form

$$\hat{H}_{AF}(t) = \hbar g \sum_i \Theta_i(t) (\hat{a}_i \hat{\sigma}_+ + \hat{a}_i^\dagger \hat{\sigma}_-), \quad (14)$$

where

$$\hbar g = d E_{\text{vac}} = d \sqrt{\frac{2\pi\hbar\omega_0}{Ac\Delta t}}.$$

This result has a clear interpretation. The traveling-wave configuration is effectively multimode, with each member of the set being a traveling packet "mode matched" to the atom. The coupling constant g depends on this mode volume. This picture is equivalent to a model of decoherence discussed by Brun [21] in which a "flying qubit" passes over a "system qubit," the former acting as an irreversible reservoir

(through its continuous spatial degrees of freedom) to carry information away from the system, thereby leading to decoherence. In our problem, a given harmonic oscillator (mode of the electromagnetic field) flies over the qubit, becomes entangled with it, and then flies away. This too leads to decoherence, as we describe in the following section.

III. THE RABI INTERACTION FOR TRAVELING-WAVES

A. Interaction with a laser beam

We consider first the case of a resonant laser beam interacting with a trapped two-level atom. The state of the field is described by a tensor product of identical coherent states for each traveling mode packet

$$|\psi\rangle_{\text{beam}} = \otimes_i |\alpha\rangle_i, \quad (15)$$

where the amplitude is given by the mean number of photons in time slice Δt , $|\alpha|^2 = P\Delta t/(\hbar\omega_0)$, with P the power of the laser beam. More precisely, the state of the beam is a statistical mixture of states of the form above, averaged over the common, but unknown, phase of the complex amplitude α , as described by Enk and Fuchs [22]. The actual value of the phase plays no role in the analysis to follow, so we choose it to be fixed with no loss of generality. In order to distinguish coherent effects from decoherence, we transform by a unitary displacement of the field states to the vacuum,

$$|\Psi\rangle \Rightarrow \hat{D}^{-1}(\{\alpha\})|\Psi\rangle, \quad \hat{A} \Rightarrow \hat{D}^{-1}(\{\alpha\})\hat{A}\hat{D}(\{\alpha\}). \quad (16)$$

In this picture,

$$\begin{aligned} H_{AF}(t) &= \hbar g \alpha (\hat{\sigma}_+ + \hat{\sigma}_-) + \hbar g \sum_i \Theta_i(t) (\hat{a}_i \hat{\sigma}_+ + \hat{a}_i^\dagger \hat{\sigma}_-) \\ &= H_{\text{coh}} + H_{\text{AV}}. \end{aligned} \quad (17)$$

The coherent term is classical Rabi oscillation at a frequency $\Omega = 2g\alpha = d\sqrt{8\pi I/\hbar^2 c}$, with I the beam intensity (cgs units, as used throughout). The second term is the atom-vacuum coupling for the traveling-wave modes only (i.e., the paraxial modes of the beam) [23].

We can now proceed with the standard Markov analysis to derive the master equation [24]. The initial atom-vacuum state is uncorrelated. After a time Δt , one of the modes becomes entangled with the atom through the atom-vacuum coupling. The Linblad (“jump”) operator \hat{L} is defined by [21]

$$\langle 1|\hat{U}_{\text{AV}}(\Delta t)|0\rangle = \frac{-i}{\hbar} \Delta t \langle 1|\hat{H}_{\text{AV}}|0\rangle = \hat{L} \sqrt{\Delta t}. \quad (18)$$

Plugging in the Hamiltonian from Eq. (17), we arrive at

$$\hat{L} = g\sqrt{\Delta t}\hat{\sigma}_- = d\sqrt{\frac{2\pi\hbar\omega_0}{Ac}}\hat{\sigma}_- \equiv \sqrt{\kappa}\hat{\sigma}_-, \quad (19)$$

where

$$\kappa = d^2 \left(\frac{2\pi k_0}{\hbar A} \right) = \Gamma \left(\frac{3\pi}{2k_0^2 A} \right) \equiv \Gamma \left(\frac{\sigma_{\text{eff}}}{A} \right)$$

is the measurement strength, Γ is the spontaneous emission rate, and σ_{eff} is the effective cross section for scattering out of the paraxial modes. By determining these jump operators, we compactly derive the master equation, equivalent to that obtained using a usual system-reservoir approach after tracing over the unmeasured bath [11,15,25],

$$\begin{aligned} \frac{d\hat{\rho}}{dt} &= \frac{-i}{\hbar} [H_{\text{coh}}, \hat{\rho}] - \frac{1}{2} \{ \hat{L}^\dagger \hat{L}, \hat{\rho} \} + \hat{L} \hat{\rho} \hat{L}^\dagger \\ &= \frac{-i}{\hbar} [H_{\text{coh}}, \hat{\rho}] - \frac{\kappa}{2} \{ \hat{\sigma}_+ \hat{\sigma}_-, \hat{\rho} \} + \kappa \hat{\sigma}_- \hat{\rho} \hat{\sigma}_+. \end{aligned} \quad (20)$$

This equation has a familiar and appealing form. It is none other than the master equation for a decaying laser-driven atom [15], but with $\Gamma \rightarrow \kappa$. The decay rate is due to the entanglement between the atom and the laser modes. Note, κ is independent of Δt , which acts as a fictitious “quantization volume” and so must be absent from any physical quantities such as the measurement strength.

The measurement strength is also independent of the laser power $|\alpha|^2$. In particular, we may turn off the laser ($\alpha \rightarrow 0$) and the measurement strength will remain the same. The ratio $\kappa/\Gamma = \sigma_{\text{eff}}/A$ may thus be interpreted as the fraction of spontaneous emission into the paraxial modes. In support of this interpretation, note that the mode area A can never be made smaller than the diffraction limit $A \sim 1/k_0^2$, so at most $\kappa \sim \Gamma$. Moreover, once the beam becomes focused to such a small spot size, one can no longer neglect the vector nature of the atom-field coupling which further decreases the measurement strength [26].

From Eq. (19), we can determine how continuous measurement by the laser beam acts to decohere the atom. For a paraxial beam, we require that $\sigma_{\text{eff}}/A \gg 1$, so that diffraction effects are minor. Then decay due to entanglement with the laser modes is small compared to decay due to spontaneous emission into 4π sr. In agreement with the conclusions of Refs. [16,17], errors in coherent control pulses due to the quantum nature of the interaction can be neglected if spontaneous emission is also negligible during the duration of the interaction.

B. Interaction with a single photon

In Sec. II, we showed how a quantization procedure in terms of nonequal-time commutators can lead to a false prediction of single-photon coherent Rabi oscillations in free space. In this section, we use our formalism to show how a quasimonochromatic and paraxial propagating single-photon wave packet drives a two-level atom.

We take the initial state of the system to be a single-photon wave packet with the atom in its ground state,

$$|\Psi(0)\rangle = \hat{a}^\dagger[f]|\text{vac}\rangle \otimes |g\rangle_A. \quad (21)$$

The operator $\hat{a}^\dagger[f] = \int dz \hat{\Psi}^\dagger(z) f(z)$ creates a delocalized single-photon state with slowly varying pulse envelope $f(z)$ [27]. For simplicity, we use a square pulse of duration $\tau = N\Delta t$. In this case, we can expand $f(z)$ in a symmetric sum of coarse-grained modes each having length $\Delta z = \Delta t/c$,

$$\hat{a}^\dagger[f] = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} \hat{a}_i^\dagger. \quad (22)$$

The state will evolve according to the Hamiltonian in Eq. (14) which commutes with the total number of excitations in the system. Neglecting, for now, the possibility of spontaneous emission into other transverse field modes, the total number of excitations will be preserved. The state at all times must then have the form

$$|\psi(t)\rangle = \left(\sum_j A_j(t) \hat{a}_j^\dagger + A_e(t) \hat{\sigma}_+ \right) |\text{vac}, g\rangle. \quad (23)$$

Consider the evolution of the system over the short interval $(t_k, t_k + \Delta t]$, where $t_k = k\Delta t, 0 \leq k \leq N-1$. We can define a map for the state between two successive time steps

$$|\psi(t_k)\rangle = e^{-i\hat{H}_k \Delta t/\hbar} |\psi(t_{k-1})\rangle, \quad (24)$$

where $\hat{H}_k = \hbar g (\hat{a}_k \hat{\sigma}_+ + \hat{a}_k^\dagger \hat{\sigma}_-)$. Using ansatz (23), we are led to the recursion relations

$$A_{j \neq k}(t_k) = A_j(t_{k-1}), \quad (25a)$$

$$A_k(t_k) = A_k(t_{k-1})c - iA_e(t_{k-1})s, \quad (25b)$$

$$A_e(t_k) = A_e(t_{k-1})c - iA_k(t_{k-1})s. \quad (25c)$$

Here, $s \equiv \sin\sqrt{\kappa\Delta t}$ and $c \equiv \cos\sqrt{\kappa\Delta t}$. The measurement strength $\kappa = g\sqrt{\Delta t}$ is the same as in Eq. (19).

These coupled algebraic equations can be solved for the amplitudes. Repeated application of the Eq. (25a) at all times $t_k < t_j$, shows that $A_k(t_{k-1}) = A_k(0) = N^{-1/2}$. Inserting this result into Eq. (25c), we have

$$A_e(t_k) = A_e(t_{k-1})c - \frac{is}{\sqrt{N}}. \quad (26)$$

This equation admits a simple series solution

$$A_e(t_k) = -\frac{is}{\sqrt{N}} \frac{1-c^k}{1-c}. \quad (27)$$

Assuming $\sqrt{N} \gg 1$, i.e., the envelope $f(z)$ is much broader than the coarse graining, we may take the limit of $A_e(t)$ as $N \rightarrow \infty$ holding $\tau = N\Delta t$. This yields

$$A_e(t) \approx -\frac{2i}{\sqrt{\kappa\tau}} [1 - e^{-\kappa t/2}]. \quad (28)$$

The solution given in Eq. (28) is based on the fundamental assumption that the evolution of the state is unitary, i.e., we consider a closed quantum system consisting of the atom and paraxial field modes. In the continuum limit, this yields an effective decay due to emission into the included paraxial modes at rate κ , but it excludes decay into all others modes which, taken together, give a total spontaneous emission rate Γ . Since we showed in Sec. III A that $\kappa \ll \Gamma$, this solution is not self-consistent. To rectify this, during the time interval Δt , we must allow for a small probability of spontaneous emission P_{spont} into nonparaxial modes. By not including these modes in our system, the state ket $|\psi\rangle$ evolves according to an effective non-Hermitian Hamiltonian [11] with decaying norm, $\langle \Psi | \Psi \rangle = 1 - P_{\text{spont}}$. Equation (25c) then reads

$$A_e(t_k) = A_e(t_{k-1})c e^{-\gamma\Delta t/2} - iA_k(t_{k-1})s, \quad (29)$$

where γ is the spontaneous emission into all nonparaxial modes. Employing the initial condition and taking the limit $\Delta t \rightarrow 0, N\Delta t = \tau$ yields

$$\frac{d}{dt} A_e(t) = -\frac{1}{2}(\gamma + \kappa)A_e(t) - i\sqrt{\frac{\kappa}{\tau}}. \quad (30)$$

Solving this equation with $\gamma=0$, using the initial condition $A_e(0)=0$, will give the same result as in Eq. (28).

Before considering the solution to this differential equation, consider the slightly altered situation in which the field starts in the vacuum state and the atom in the excited state. In this case, it is easy to see that the second term on the right side of Eq. (30) will vanish. Then, the equation becomes

$$\frac{d}{dt} A_e(t) = -\frac{1}{2}(\gamma + \kappa)A_e(t), \quad (31)$$

with the initial condition $A_e(0)=1$. This must give the standard exponential decay due to spontaneous emission in the vacuum

$$A_e(t) = e^{-\Gamma t/2}. \quad (32)$$

This allows us to equate $\Gamma = \gamma + \kappa$, where κ is again seen to be the spontaneous emission rate due to the contribution of the modes in the paraxial beam.

The general solution to Eq. (30) is

$$A_e(t) = -\frac{2i}{\Gamma} \sqrt{\frac{\kappa}{\tau}} (1 - e^{-\Gamma t/2}), \quad (33)$$

holding for times $0 \leq t \leq \tau$. The probability of the atom being in the excited state is then,

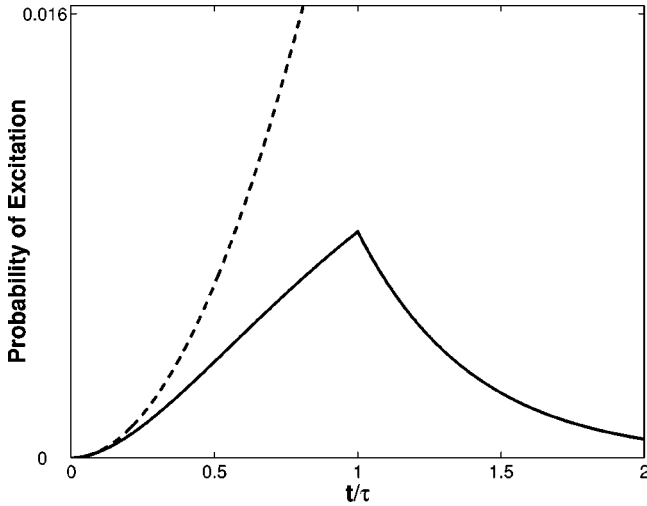


FIG. 2. The probability for a single photon in free space to excite a two-level atom (solid line) differs significantly from the standard Rabi solution, in which an atom is coupled to a photon in a single-mode cavity (dashed line). The parameters used here are $\Gamma\tau=2.5$ and $\kappa/\Gamma=1/50$ (see text).

$$P_e(t) = 4 \frac{\kappa}{\Gamma^2 \tau} (1 - e^{-\Gamma t/2})^2 \quad (34)$$

during the same interval. This is a monotonically increasing function of t , and so achieves its maximum at the upper limit $t = \tau$, after which the excitation probability can only decay (Fig. 2). Then, the maximum probability for any τ can be found by solving

$$\frac{d}{d\tau} P_e(\tau) = 0 \rightarrow \Gamma\tau \approx 2.5. \quad (35)$$

At this point, the probability is $P_e \approx 0.8\kappa/\Gamma$, which is necessarily less than 1 given that $\kappa/\Gamma < 1$ as previously discussed. Thus, in the paraxial approximation, no single-photon pulse can be constructed that excites an atom with certainty.

This result is, of course, not surprising. Symmetry ensures that the only single-photon pulse capable of exciting an atom with unit probability is the time reversal of a spontaneously emitted packet [28]. Such a pulse represents the vector spherical harmonic associated with the atomic radiation pattern (assuming emission on a transition with well-defined angular momentum), a field not captured in the scalar paraxial approximation. Moreover, even including the vector nature of the field beyond paraxial will not be sufficient to yield high excitation probability. The field must be well “mode matched” to the atom’s radiation pattern to give strong coupling between the atom and a single photon in free space [26].

C. Interaction with a large n -photon Fock pulse

In the last section, we showed that in free space the coupling of a single photon to an atom will not lead to coherent Rabi oscillations. In contrast, for an n -photon Fock state with

a large n , we expect the system to be dominated by stimulated emission. A rigorous treatment of this problem using the Bethe-Ansatz was given by Rupasov and Yudson [29]. We show here how this phenomenon is recovered simply in the present formalism.

In general, an arbitrary division may be envisioned in Hilbert space that separates system from reservoir. In Sec. II, the system was chosen to be the paraxial field modes plus the atom. Alternatively, we may take the system to consist of the atom interacting with the single pulse mode defined by creation operator $\hat{a}^\dagger[f]$ in Eq. (22), with all other modes making up the environment. Such a choice can always be made, and cannot change the physics. We may then choose to ignore all environmental modes if $\Gamma\tau \ll 1$, with the caveat that any effects that occur in the system must have scales much larger than $\Gamma\tau$ in order to be considered valid.

Consider a pulse of length τ such that $\Gamma\tau \ll 1$. Further, take the state of the system to only have excitations in this pulse mode such that

$$|\psi(0)\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger[f])^n \otimes |0, g\rangle = |n, g\rangle. \quad (36)$$

The rest of the modes are in the vacuum state and are treated as an environment. Ignoring terms of the order of $\Gamma\tau$, the system evolves under the single-mode Hamiltonian

$$\hat{H} = \hbar g_{\text{eff}} (\hat{a}[f] \hat{\sigma}_+ + \hat{a}^\dagger[f] \hat{\sigma}_-). \quad (37)$$

This is none other than the Jaynes-Cummings Hamiltonian restricted to an initial manifold with n excitations and having effective coupling constant $\hbar g_{\text{eff}} = d \sqrt{2\pi\hbar\omega_0/V_{\text{pulse}}}$, where $V_{\text{pulse}} = Ac\tau$ is the pulse volume. The system undergoes the familiar coherent Rabi oscillations within the two-dimensional manifold, as in Eq. (2),

$$|\psi(t)\rangle = \cos(g_{\text{eff}}\sqrt{n}t) |g\rangle |n\rangle - i \sin(g_{\text{eff}}\sqrt{n}t) |e\rangle |n-1\rangle. \quad (38)$$

This solution applies to the single photon case as well. The probability amplitude from Eq. (33) limits to

$$A_e(t) \rightarrow -i g_{\text{eff}} t [1 + O(\Gamma t)], \quad (39)$$

which is the correct limit of sinusoidal Rabi oscillation. However, during the pulse duration not even a single oscillation can occur since $g_{\text{eff}} = \sqrt{\kappa/\tau} \ll 1/\tau$. Thus, for true oscillations to occur, one must have

$$g_{\text{eff}} \sqrt{n} \gtrsim \frac{1}{\tau} \quad \text{or} \quad n\Gamma\tau \gtrsim \frac{A}{\sigma_{\text{eff}}}. \quad (40)$$

The last inequality can be interpreted as saying that the mean number of photons emitted via stimulated emission into the pulse must dominate over spontaneous emission, even when the spontaneous photons are paraxial. When these conditions hold we may *consistently* ignore all initially unoccupied

modes, yet still recover dynamics in agreement with the usual Jaynes-Cummings Hamiltonian. The multimode description of the field becomes superfluous.

Finally, what is the measurement strength associated with probing an atom using a large photon number Fock-state pulse? Unlike the coherent state case, the field does not factorize into uncorrelated temporal slices. In fact, when viewed in terms of the coarse-grained modes, the Fock pulse in highly *entangled*. Detection of photons at the leading edge of the pulse will introduce new fluctuations in the trailing edge, which has yet to interact with the atom. This implies that the usual Markov approximations do not hold. The Fock pulse is most naturally treated as part of the “system” rather than an “environment” which continuously carries information away from the atom.

IV. QND MEASUREMENT VIA FARADAY POLARIZATION SPECTROSCOPY

The resonant interaction considered up to this point, though fundamental in nature, has little practical application to the problem of continuous measurement in free space since the measurement strength is always bounded from above by the spontaneous emission rate. We thus turn our attention to an off-resonant interaction. In particular, we consider the problem of measuring a spin component of an atom through the Faraday effect wherein the linear polarization of a probe laser rotates by an amount proportional to the magnetization of the sample. This interaction provides for a “quantum nondemolition measurement” (QND) of the atom, i.e., the probe measures an observable (here, the atomic spin along the laser propagation direction) without perturbing that observable [30,31]. Such an interaction has been applied to ensembles of atoms to produce spin-squeezed states [13] and to demonstrate entanglement between two spatially separated ensembles [14].

We review here the basic physics associated with this measurement scheme. The interaction energy is that of an electric-polarizable particle in a vector field whose Hamiltonian may be written as

$$\hat{H}_{\text{int}} = -\hat{\mathbf{E}}^{(-)} \cdot \hat{\alpha} \cdot \hat{\mathbf{E}}^{(+)}, \quad (41)$$

where $\hat{\alpha}$ is the atomic polarizability tensor and \mathbf{E} is the complex electric-field amplitude. Expressing this equation in terms of irreducible tensor components, the interaction can be decomposed into an effective scalar, vector, and symmetric rank-2 contribution. We consider here the case of alkali atoms probed on the so-called $D2$ line $S_{1/2} \rightarrow P_{3/2}$, for which the ground-state atomic polarizability operator is [32]

$$\vec{\alpha} = \alpha_{\text{lin}} \left(\hat{1} + \frac{1}{2} (\mathbf{e}_+ \mathbf{e}_+^* - \mathbf{e}_- \mathbf{e}_-^*) \hat{\sigma}_z \right). \quad (42)$$

Here, \mathbf{e}_{\pm} are the right and left helicity vectors relative to the quantization axis along the probe propagation direction, $\hat{\sigma}_z = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$ is the Pauli spin operator for the ground-state electron, and α_{lin} is the atomic polarizability for fields

with linear polarization. This equation has a simple interpretation. The first “scalar term” gives rise to an effect that depends solely on the field intensity, whereas the term proportional to $\hat{\sigma}_z$ depends on the field ellipticity. Clearly, the irreducible rank-2 component vanishes here [32–34].

Under the paraxial approximation, we take the field to be approximately a plane wave with two polarization vectors. The quantum field associated with the complex amplitude is

$$\hat{\mathbf{E}}^{(+)} = \sqrt{\frac{2\pi\hbar\omega}{V}} (\hat{a}_- \mathbf{e}_- + \hat{a}_+ \mathbf{e}_+) e^{ikz}, \quad (43)$$

where V is the effective quantization volume for the propagating mode. Substituting this into Eq. (41), the quantum Hamiltonian becomes

$$\hat{H}_{\text{int}} = -\frac{2\pi\alpha_{\text{lin}}}{V} \hbar\omega \left[(\hat{N}_+ + \hat{N}_-) + \frac{1}{2} (\hat{N}_+ - \hat{N}_-) \hat{\sigma}_z \right], \quad (44)$$

where \hat{N}_{\pm} is the number operator for photons in the positive or negative helicity states. The scalar term gives rise to an overall phase shift (index of refraction) and thus, can be absorbed into the free field Hamiltonian. The vector term gives rise to the Faraday effect. Recognizing $\hat{J}_z = (\hat{N}_+ - \hat{N}_-)/2$ as the total field helicity, the effective interaction Hamiltonian takes the QND form

$$\hat{H}_{\text{int}} = -\frac{2\pi\alpha_{\text{lin}}}{V} \hbar\omega \hat{J}_z \hat{\sigma}_z. \quad (45)$$

Under this Hamiltonian, the photon spin becomes correlated with the atom’s magnetic moment and thus the laser polarization may act as a meter for the atomic spin. Since $\hat{\sigma}_z$ also commutes with the system-meter Hamiltonian, it is clearly a QND variable [15].

Our Hamiltonian still contains the undefined quantization volume V . To rectify this, we follow the formalism introduced in Sec. II. Introducing propagating modes of duration Δt , so that $V \rightarrow A c \Delta t$, the Hamiltonian becomes

$$\hat{H}_{\text{int}} = -\sum_i 2 \frac{\hbar\chi}{\Delta t} \Theta_i(t) \hat{J}_z \hat{\sigma}_z, \quad (46)$$

where $\chi = \pi\alpha_{\text{lin}}\omega/(cA) \approx (\sigma_0/A)[\Gamma/(-2\Delta)]$. Here, Δ is the (far) detuning from the atomic resonance with linewidth Γ and σ_0 is the on-resonance absorption cross section for linear polarization.

With the Hamiltonian so defined, the evolution of the system may be calculated. The system shall consist of an atom interacting with a laser beam that is linearly polarized in the x direction. The initial state of the laser is then

$$|\Phi\rangle_{\text{probe}} = \otimes_i \frac{1}{\sqrt{2}} (|\alpha\rangle_{ix} |0\rangle_{iy}), \quad (47)$$

where x and y label the two orthogonal linear polarization modes. As in Sec. III A, $|\alpha|^2 = P\Delta t/\hbar\omega_0$, with P the power in the beam. Given this initial state for the field, we may define the marginal density operator for the atom alone by tracing over the field at time t . Since Hamiltonian (41) only couples the atom to the k th field mode in time interval $(t_k, t_k + \Delta t]$, the reduced atomic state evolves during this interval as

$$\hat{\rho}(t_k + \Delta t) = \text{Tr}_k[\hat{U}_i|\alpha\rangle_{ix}\langle\alpha|_{ix}\otimes|0\rangle_{iy}\langle 0|_{iy}\otimes\hat{\rho}(t_k)\hat{U}_i^\dagger]. \quad (48)$$

In the interaction picture, the unitary evolution becomes

$$\begin{aligned} \hat{U}_i|\alpha\rangle_{ix}|0\rangle_{iy} &= e^{-i\chi\hat{\sigma}_z(\hat{N}_{+i}-\hat{N}_{-i})}|\alpha\rangle_{ix}|0\rangle_{iy}, \\ &= |\alpha\cos(\chi\hat{\sigma}_z)\rangle_{xi} - \alpha\sin(\chi\hat{\sigma}_z)\rangle_{yi}. \end{aligned} \quad (49)$$

This expression should be interpreted in the context of a matrix element, given that atomic operators appear in the labels for the field kets. Clearly, the field and atomic states will generally become entangled by the interaction.

The map in Eq. (48) constitutes a continuous measurement on the field. To see this explicitly, we must expand in powers of Δt , and since $|\alpha|^2 \propto \Delta t$, this is equivalent to an expansion of the state kets in Eq. (49) in powers of α . In the Fock (photon number) basis, we have

$$\begin{aligned} \hat{U}_i|\alpha\rangle_{ix}|0\rangle_{iy} &\approx \left(1 - \frac{|\alpha|^2}{2}\right)[|0\rangle_{xi}|0\rangle_{yi} + \alpha\cos(\chi\hat{\sigma}_z)|1\rangle_{xi}|0\rangle_{yi} \\ &\quad - \alpha\sin(\chi\hat{\sigma}_z)|0\rangle_{xi}|1\rangle_{yi}]. \end{aligned} \quad (50)$$

Substituting this back into Eq. (48), one finds

$$\begin{aligned} \hat{\rho}(t_k + \Delta t) &= \rho(t_k) + |\alpha|^2[-\rho(t_k) + \sin(\chi\hat{\sigma}_z)\rho(t_k)\sin(\chi\hat{\sigma}_z) \\ &\quad + \cos(\chi\hat{\sigma}_z)\rho(t_k)\cos(\chi\hat{\sigma}_z)]. \end{aligned} \quad (51)$$

This form may be further simplified since $\chi \ll 1$ for a single atom probed by a far off-resonance laser. Taking the limit, $\Delta t \rightarrow 0$ and keeping terms to second order in χ gives the master equation

$$\frac{d\hat{\rho}}{dt} = \frac{P\chi^2}{\hbar\omega_0} \left[\hat{\sigma}_z\hat{\rho}\hat{\sigma}_z - \frac{1}{2}\{\hat{\sigma}_z^2, \hat{\rho}\} \right] = -\frac{\kappa}{2}[\hat{\sigma}_z, [\hat{\sigma}_z, \hat{\rho}]], \quad (52)$$

where the measurement strength can be easily identified from the familiar Linblad form of the master equation [11] as

$$\kappa = \frac{P\chi^2}{\hbar\omega_0}. \quad (53)$$

Note that the steady-state solutions to Eq. (52) are the Dicke states, so this technique provides a QND measurement of

spin. This expression was also derived by Thomsen and Wiseman in the context of control of atom-laser coherence [35].

The physics of the continuous measurement under consideration here differs substantially from that of the resonant Jaynes-Cummings interaction studied in Sec. III. Here, the measurement strength depends upon the power in the beam, whereas the previous result had no such dependence. For the resonant case, κ could be explained as arising from those spontaneous photons emitted into paraxial modes. In contrast, the dependence of the QND measurement on the laser power indicates that the measurement strength is due to the coherent redistribution of photons between the polarization modes in a manner depending on the atomic state.

V. SUMMARY AND DISCUSSION

Continuous quantum measurement, once studied only in gedanken experiments, can now be realized in laboratory applications [8,36]. Such applications utilize the ability to continuously gather information from a probe coupled to a quantum system. The system state then evolves under a stochastic master equation, characterized by the measurement strength. In this paper, we examined how one may derive the measurement strength for paraxial laser probes in free space by considering two examples.

We considered first the fundamental system of an atom coupled to a laser beam. Classically the atom-laser interaction leads to Rabi oscillation which is often used to manipulate coherent superpositions of atomic states. In particular, laser control pulses can be used to implement quantum logic gates [37]. Quantum mechanically continuous measurement of the atom by the quantum laser pulse can lead to entanglement between the system and the probe, inducing errors in logic gate operation. We found that this is not a concern as the measurement strength is bounded from above by the total spontaneous emission rate and can thus be neglected for short interaction times. In fact, the measurement rate may be attributed to spontaneous emission into the paraxial modes, which will always be small compared to emission into 4π sr with the appropriate vector field. We also studied the distinction between continuous observation by a coherent beam and by a pulse with well-defined photon number. In the latter case, correlations between the quantum fluctuations at different points in the pulse disallow a clean separation between system and probe. Instead, the entire atom-field system will coherently oscillate if the number of photons is sufficiently large. For a single photon, however, we showed that Rabi oscillations cannot occur in free space. Such a result is essential lest a distant detector be able to instantly sense the presence of the atom. This highlights the need to treat the propagation of quantum fluctuations in a traveling-wave geometry with great care to avoid contradictions with causality.

Another mechanism allowing for continuous measurement of an atom is the Faraday rotation of an off-resonant polarized beam, induced by the atom's magnetic moment. This interaction produces a QND measurement of the atomic spin. The measurement is due to stimulated redistribution of photons between pairs of occupied modes, in contrast to the

resonant Rabi interaction where measurement results from spontaneous emission. Since the measurement strength is proportional to the laser intensity, it can be made much larger than the spontaneous emission rate. Of course, spontaneous emission is not the only source of noise. In order to see the effects of quantum back action, the quantum “projection” noise [38] must be large compared to all other noise sources, such as those due to photodetection. This implies that an experiment in which the back action is important must enhance the coupling constant without increasing the shot noise, possibly through the use of an optical cavity or by maintaining a large ensemble of atoms collectively coupled to the probe. The latter approach has led to the observation of spin squeezing [13] and ensemble entanglement [14]. In such a situation, it would be interesting to understand the effective measurement strength by the probe on any one member of

the ensemble. Exploring this question will require us to consider how quantum correlations are shared within a multipartite system interacting with a probe. We plan to address this question in future work.

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