Classification of Lie groups with simple lie algebra

- Last time we discussed simple groups & simple lie algebras

Recap: * Simple $G \equiv$ No nontrivial $N \triangleleft G$. Normal subgroup

* Simple $\mathfrak{g} \equiv$ No nontrivial $\mathfrak{n} \triangleleft \mathfrak{g}$. Ideal subalg.

- We ended with the statement that,

* Simple $\mathfrak{g} \not\Rightarrow$ Simple $G = e^{\mathfrak{g}}$

ex. $SU(2)$: $\langle i\sigma_x, i\sigma_y \rangle = \mathfrak{su}(2)$ is simple

but $\{\mathbb{1}, -\mathbb{1}\} \triangleleft SU(2)$ not simple!

* However Simple $G = e^{\mathfrak{g}} \Rightarrow$ Simple $\mathfrak{g}$.

Pf: (Contrapositive) S.p. $\mathfrak{g}$ not simple. Then $\exists\, \mathfrak{n} \triangleleft \mathfrak{g}$.

Let $h \in \mathfrak{g}$ and $m \in \mathfrak{n}$. Then

$$e^{h} e^{m} e^{-h} = e^{e^{h} m e^{-h}} = e^{m + \overset{\in \mathfrak{n}}{[h,m]} + \overset{\in \mathfrak{n}}{[h,[h,m]]} + \dots}$$

Since $m \in \mathfrak{n}$ $ad_h^{(k)}(m) \in \mathfrak{n}$. So $e^{\mathfrak{n}}$ is a normal subgroup! Thus G is not simple. □

Remark: This is part of a general theme that,

Lie Subalgebras $\Longleftrightarrow$ Connected Lie Subgroups

$$\begin{tikzcd} \langle i\sigma_z \rangle \arrow[d, "\exp"'] & < & \mathfrak{su}(2) \arrow[d, "\exp"] \\ U(1) & < & SU(2) \end{tikzcd}$$

ex.

Lie Alg. Ideals $\Longleftrightarrow$ Connected Normal Lie Subgroups

$\mathfrak{g} = \langle i\mathbb{1} \rangle \oplus \langle iQ, iP \rangle \oplus \langle i(Q^2+P^2), i(QP-PQ) \rangle$?

- So what does simple $\mathfrak{g}$ get us?

$$\begin{tikzcd} \text{simple } \mathfrak{g} \arrow[r, Rightarrow, "(b)"] \arrow[d, Rightarrow, "(a)"'] & N \triangleleft e^{\mathfrak{g}} \text{ is discrete} \arrow[d] \\ \text{$G=e^{\mathfrak{g}}$ path connected} \arrow[r] & \& \arrow[d, Rightarrow, "(c)"] \\ & N \triangleleft e^{\mathfrak{g}} \text{ is abelian and central} \end{tikzcd}$$

Not Proven here: If $e^{\mathfrak{g}}$ is compact N is finite.

Pf (a): Easy, if I have elements in the alg $\{X_\mu\}$, e^{X_μ} is path connected to the identity by $\{e^{\theta^\mu X_\mu} \mid \text{each } \theta_\mu \in [0,1]\}$.

Pf (b): (Contradiction) Suppose $N \triangleleft e^{\mathfrak{g}}$ is continuous. Then $N = e^{\mathfrak{n}}$ Consider an element $n_1 \in \mathfrak{n}$. Then, for some $h \in \mathfrak{g}$, $e^{h} e^{n} e^{-h} = e^{n_1 + [h, n_1] + \dots}$. If $Ad_h^{k}(n_1) \in \mathfrak{n}$ this would imply $\mathfrak{g}$ is not simple! So $Ad_h(n_1) = 0$. Thus each element of N is distinct. Thus N is discrete.

Pf (c): a&b are true for any $g \in G$ so $gng^{-1} = n$ $\forall n \in N$. Thus, $gn = ng$ so N is abelian (choose $g \in N$) and central (Commutes with every element of G).

- Thus we now have a better understanding of which Lie groups G have the same simple lie algebra $\mathfrak{g}$.
  - Given lie group G with central subgroup N we can construct another lie group G' with the same lie algebra $\mathfrak{g}$ by the sequence of maps,

$$N \overset{\iota}{\hookrightarrow} G \xrightarrow{\pi} G' \quad \text{where } \operatorname{Im}\iota = \operatorname{Ker}\pi.$$

* Such a sequence of maps is a short-exact sequence.
* In the context of groups, we say G is an extension of G' by N.
* Since $N < Z(G)$ we call this a central extension.
* This also means $G/N \simeq G'$.

- A well known example of this in mathematical physics is the double-covering of $SO(3)$ by $SU(2)$.

Remark 1: $\mathfrak{su}(2) = \langle i\sigma_x, i\sigma_y \rangle \simeq \mathfrak{so}(3) = \langle iJ_x, iJ_y \rangle$

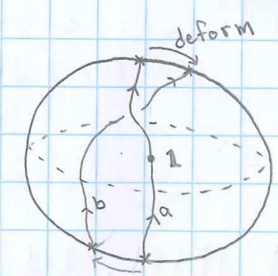
So the have the same Lie algebra.

Remark 2: $SO(3)$ is not simply connected.

→ To visualize an $SO(3)$ transformation notice we need a direction (unit vector $\bar{n}$) and an angle θ.

→ Since $R_{\bar{n}}(\pi) = R_{\bar{n}}(-\pi)$ we can visualize $SO(3)$ as a ball of radius π embedded in $\mathbb{R}^3$ where anti-podal points on the surface are identified with eachother.

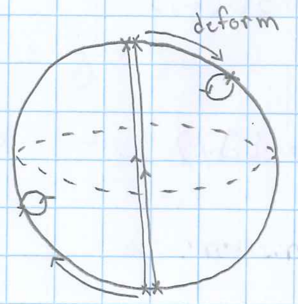
→ Consider the following loop in $SO(3)$



* by deforming path a we get path b, one can see path a cannot be continuously deformed to a point!

→ Thus $SO(3)$ is not simply connected!

→ If instead we consider a different path,



* We see a loop that traverses the ball twice can be contracted to a point.

- It turns out that there are only two types of loops in $SO(3)$. Another way to say this is the <u>fundamental group</u> is $\pi_1(SO(3)) = \mathbb{Z}_2$.

- The additional "hole" in $SO(3)$ is exactly related to the 2-1 mapping from $SU(2)$ to $SO(3)$.

$$\mathbb{Z}_2 \xhookrightarrow{i} SU(2) \xrightarrow{\pi} SO(3)$$

$$\text{or} \quad SU(2)/\mathbb{Z}_2 \simeq SO(3)$$

- What is the map π?

→ Let $U \in SU(2)$. Let $\bar{n} \in \mathbb{R}^3$ ∋ $|\bar{n}| = 1$. Let $\bar{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$

The map π takes $U \in SU(2)$ and maps it to $R \in SO(3)$ such that,

$$U(\bar{n}\cdot\bar{\sigma})U^{+} = (R\bar{n})\cdot\bar{\sigma}$$

→ This map is 2-1 since $\pm U \xmapsto{\pi} R$.

→ Notice $\operatorname{Ker}\pi = \{\mathbb{1}, -\mathbb{1}\} \trianglelefteq SU(2)$. So by the first isomorphism theorem $SU(2)/\mathbb{Z}_2 \simeq SO(3)$.

- This is the spirit of the general theme we wish to follow in constructing Lie groups with the same simple Lie algebra.

- To classify all Lie groups with the same simple lie algebra we first must find the largest such Lie group (<u>Universal Cover</u> $\hat{G}$).

* The universal cover is always simply connected and has the largest center.

- Lets describe this!

Lie Group	Cover	Center
$A_n = SL(n+1, \mathbb{C})$	$SL(n+1, \mathbb{C})$	$\mathbb{Z}_{n+1}$
$B_n = SO(2n+1, \mathbb{C})$	$Spin(2n+1, \mathbb{C})$	$\mathbb{Z}_2$
$C_n = Sp(2n)$	$Sp(2n)$	$\mathbb{Z}_2$
$D_n = SO(2n, \mathbb{C})$	$Spin(2n, \mathbb{C})$	$\begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 & 2n = 0 \bmod 4 \\ \mathbb{Z}_4 & 2n = 2 \bmod 4 \end{cases}$

- A quick note on Spin(n).

  * Spin(n) is generated by a quadratic subalgebra of the Clifford algebra on n-dimensional euclidean space.

    ie $\mathrm{Spin}(n) = \left\{ e^{\frac{1}{2}A^{\alpha\beta}\gamma_\alpha\gamma_\beta} \;\middle|\; \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \right\}$

  * Spin(n) double covers SO(n).

    $$\mathbb{Z}_2 \xhookrightarrow{i} \mathrm{Spin}(n) \xrightarrow{\pi} SO(n)$$

    π acts by taking $U \in \mathrm{spin}(n)$ and mapping it to $R \in SO(n)$ such that, for $U = e^{\frac{1}{4}A^{\alpha\beta}\gamma_\alpha\gamma_\beta}$

    $$U\, \vec{n}\cdot\vec{\gamma}\, U^{-1} = (R\vec{n})\cdot\vec{\gamma}.$$ It turns out,

    $$R = e^{A}.$$

- Finally, one can use all of this information to construct classification tables,

Real Forms

A_3	$SL(4,\mathbb{C})$	$SL(4,\mathbb{R})$	$SU(4)$	$SU(3,1)$	$SU(2,2)$	$SU(4)^*$
$\mathbb{Z}_1$	$SL(4,\mathbb{C})$	ect.				
$\mathbb{Z}_2$	$SL(4,\mathbb{C})/\mathbb{Z}_2$					
$\mathbb{Z}_4$	$SL(4,\mathbb{C})/\mathbb{Z}_4$					

Central Subgroups

adjoint rep!

- As a final note, one may be puzzled as to why we said SU(2) double-covers SO(3). In fact it is also the universal cover of SO(3). But shouldn't this be Spin(3)? What is going on?

- It turns out $\mathfrak{spin}(3) \simeq \mathfrak{su}(2)$ purely by accident. This is not a general trend.

- There are five <u>accidental Lie algebra isomorphisms</u>.

  $\mathfrak{so}(3) \simeq \mathfrak{su}(2) \simeq \mathfrak{sp}(2) \longrightarrow A_1 \simeq B_1 \simeq C_1$

  $\mathfrak{sp}(4) \simeq \mathfrak{so}(5) \longrightarrow B_2 \simeq C_2$

  $\mathfrak{so}(2)$ is simple $\longrightarrow D_1 \simeq U(1)$

  $\mathfrak{so}(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2) \longrightarrow D_2 \simeq A_1 \otimes A_1$

  $\mathfrak{so}(6) \simeq \mathfrak{su}(4) \simeq D_3 \simeq A_3$

- To avoid the accidents we usually enumerate the lie groups as,

  $1 \leq n: A_n$
  $2 \leq n: B_n$
  $3 \leq n: C_n$
  $4 \leq n: D_n$.