PHYC 500.001 Adv Sem: “Lie groups in quantum information”

Course coordinators:

- Christopher S. Jackson (chrisjackson@unm.edu)
- Pablo M. Poggi (ppoggi@unm.edu)

Course duration: June 5^{th} – July 31^{st} (nine sessions in total). Note: there will be no class on June 19^{th}. We will schedule a make-up session in the first day.

Format: This course is a survey of Lie groups and their applications to physics and quantum information. The course is designed to be a ‘self-study’ program. In the first two sessions, the course coordinators will give introductory lectures on Lie groups and Lie algebras. Reading material will be assigned every week, including lecture notes and research articles. In the remaining sessions, course participants will present sections from the reading material and facilitate group discussions. Participants are encouraged to also propose other topics of their interest.

Prerequisites: This course assumes basic familiarity with groups and quantum mechanics at the level of Physics 521-522.

Assignments: All participants are required to read the materials assigned for each session and participate in group discussions. Participants should be willing to present assigned material in later sessions.

Syllabus:

Pablo and Chris will organize the first two sessions

Session 1 (June 5th): **Groups and symmetry**—groups, finite groups, matrix groups, and the classification theorems for simple groups (finite and finite-dimensional matrix.)

Examples and exercises.

Session 2 (June 12th): **Lie algebras and analyticity**—the exponential map, the adjoint representation, the Baker-Campbell-Haussdorf formula, and Hilbert's 5th problem.

Examples and exercises.

Topics to be developed by course participants in the remaining sessions

1) **Session 3**: **Global classification of semisimple Lie groups—**Universal covers and the accidental isomorphims, su(2)=sp(2)≅so(3), so(4)≅su(2)⊕su(2), so(6)≅su(4), and so(5)≅sp(4). Also su(1,1)≅sp(2,ℝ)=so(2,1).

   Application: A comment on the standard model. More applications in sessions 6 and 7.
2) **Session 4**: **Semidirect products and normality**—Normal subgroups, conjugacy classes, semidirect products, automorphisms, and cosets.

   Application: the Pauli and Clifford groups and quantum error correction.
3) **Session 5**: **Introduction to Quantum control—**Reachable sets and dynamical Lie algebras.

   Application: So many to choose from.
4) **Session 6**: **KAK decompositions and singularity**—Cartan subalgebras, Cartan involutions, the Weyl group, the Weyl chamber, rank, and Riemann duality.

   Application: SU(4) and the control of two qubits.
5) **Session 7**: **Non-compact semisimple Lie groups**—the Iwasawa decomposition, Gaussian decomposition, and bounded domains.

   Application: Sp(2n,ℝ) and bosons.
6) **Session 8**: **Representation theory I**—Linear representations, invariant subspaces, irreducible representations, Schur's lemma, representations of permutation, and representations of SU(N), and Young Tableaux.

   Application: Wigner-Eckhart theorem.
7) **Session 9: Representation theory II**—.Linear representations, invariant subspaces, irreducible representations, Schur's lemma, representations of permutation, and representations of SU(N), and Young Tableaux.

   Application: Shur-Weyl duality

Additional topics in case there is significant interest

Topic A: **Measure theory and invariance**—forms, wedge products, the pullback, the Haar measure, and the Weyl integration trick.

Application: Random-matrix theory.

Topic B: **Invariant theory and symmetry**—enveloping algebras, Casimir operators, polynomial rings, and Reynolds projection.

Application: generalized purity, and generalized entanglement.

Topic C: **Projective representations of Abelian groups**—Group cohomology.

Application: SPTO phase classification.

Topic D: **Wavelet theory**—The affine group.

Application: Renormalization and JPEG files.

Topic E: **Analytic signaling theory**—The Weyl-Heisenberg group.

Application: Time-frequency localization and audio signal storage.

Some general references:

- B. Hall – *Lie Groups, Lie Algebras and Representations: An Elementary Introduction.* Springer – Graduate Texts in Mathematics. Introduces most concepts and properties related to Lie groups and algebras using matrix groups, which makes it a very accessible to read.
- W. Fulton and J. Harris – *Representation Theory: A First Course*. Springer – Graduate Texts in Mathematics. Explicitly and individually discusses SU(2) through SU(4), SO(3) through SO(8), and Sp(2) through Sp(6).
- W. Tung – *Group Theory in Physics*. World Scientific Publishing. This was Chris' first ever read on group theory!
- A. Barut and R. Raczka – *Theory of Group Representations and Applications*. Polish Scientific Publishers. This is a comprehensive survey of Lie group theory, and contains citations to most original works.

Dictionary for Lie groups:

We want to construct a list of notions and properties typically associated with Lie groups, whose definitions would be nice to have at hand as a sort of "dictionary"

- E.g.: connected, compact, solvable, semisimple, etc.
- We will build the dictionary as we go forward in the course, with each presenter filling the appropriate entry.