An introduction to groups by Chris Jackson 5th of June, 2019

- A group is a set G equipped with a "multiplication" rule such that G contains an identity ($\exists e, \forall g : eg = g$) and inverses ($\forall g, \exists g^{-1} : g^{-1}g = e$.) Further $\forall g, h, k : (gh)k = g(hk)$.

- The spirit of this definition I would describe by this reflection: A set is defined by imagining a property and multiplication provides a dynamic amoung the elements of a set.

- Usually, this principle is ~~described as~~ called "symmetry." Hermann Weyl liked to use the word "relativity." Emmy Noether would use a term like "invariance."

- If ever you arrive at a sense of complete mathematical comprehension about a specific phenomenon, I would guess that that sense is because your activity rests on this principle and thus a group could help you to articulate your sense. ☺

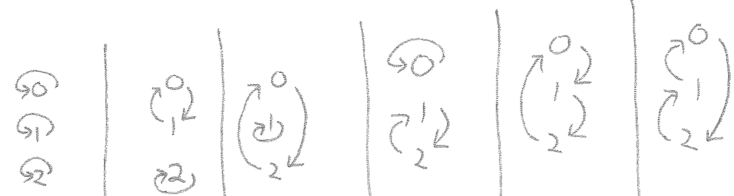
• Abstract group multiplication is equivalent to the composition of invertible maps.

The Symmetric groups

$S_N = \{\text{invertible maps/mutations amoung } N \text{ elements}\}$

$= \{\text{"permutations"}\}$

E.g. S_3:



Cayley's cyclic notation:	e	(01)	(02)	(12)	(012)	(021)

~~The~~ Notice $(01)^2 = e$ and $(012)^3 = e$, where $g^n = \overbrace{gg\cdots g}^{n \text{ times}}$.

The Cyclic groups

$\mathbb{Z}_N = \{a, a^2, a^3, \ldots, a^N = e\}$

Noncommutativity a.k.a. NonAbelianity

A group G is Abelian if $\forall g, h \in G : gh = hg$.

E.g. In S_3, $(012)(01) = (02) \neq (12) = (01)(012)$.

~~In general, $\pi^{-1}(a_1 \cdots a_k)\pi = (a$~~ Equivalently, $(012)(01)(012)^{-1} = (12)$, etc.

In general $\pi(n_0 \cdots n_k)\pi^{-1} = \big(\pi(n_0) \cdots \pi(n_k)\big)$.

The classification of simple finite ~~simple~~ groups

Roughly ~~Loosely~~ speaking (we will clean this up as the course continues,) there are only 4 basic classes of finite groups:

1) The cyclic,
2) The symmetric,
3) Those of Lie type, and
4) The 27 sporadic.

The third class is in a sense the most diverse and is related to the theory of Lie groups.

Lie groups

Roughly speaking, a Lie group is a continuous group. We will discuss the nature of Lie groups much more deeply as the course continues, but for now it suffices to say that finite-dimensional Lie groups are equivalent to matrix groups.

Matrix groups

Let $\mathbb{C}$ be the (algebraically closed field of) complex numbers and $\mathbb{R}$ be the real numbers (of which $\mathbb{C}$ is a finite field extension.)

The General Linear group

$$GL(n,\mathbb{C}) = \{\text{invertible } n\times n \text{ matrices with entries in } \mathbb{C}\},$$

equipped with matrix multiplication.

E.g. $GL(1,\mathbb{C}) = \mathbb{C}\setminus\{0\} \equiv \mathbb{C}_0$

This group is abelian and has two "real forms":

1) The circle, $\mathcal{U}(1) = \{z\in\mathbb{C}_0 : z^{-*} \equiv (z^{-1})^* = z\}$ and

2) The line, $e^{\mathbb{R}} = \{z\in\mathbb{C}_0 : \exists x\in\mathbb{R},\ z = e^x\}$.

The determinant

A very important map is $\det : GL(n,\mathbb{C}) \twoheadrightarrow \mathbb{C}_0$ ~~such that~~

such that

$$\det(M) = \frac{1}{n!}\,\varepsilon_{i_1\cdots i_n} M^{i_1}{}_{j_1}\cdots M^{i_n}{}_{j_n}\,\varepsilon^{j_1\cdots j_n}$$

where $\varepsilon_\pi = (-1)^\pi$ is the sign of the permutation $\pi(k) = i_k$.

The double arrow means the map is onto and a group homomorphism — that is $\det(MN) = (\det M)(\det N)$

The Special Linear groups

$$SL(n,\mathbb{C}) = \{M \in GL(n,\mathbb{C}) : \det M = 1\} = \det^{-1}(1) = \ker(\det).$$

Real forms

$$SL(n,\mathbb{R}) = \{M \in SL(n,\mathbb{C}) : M^* = M\}.$$

$$SU(n) = \{M \in SL(n,\mathbb{C}) : M^{-\dagger} = M\}.$$

$$SU(r,s) = \{M \in SL(n,\mathbb{C}) : g_{rs}^{-1} M^{-\dagger} g_{rs} = M\}.$$

$$\text{where } g_{rs} = \begin{bmatrix} 1_{r\times r} & 0 \\ 0 & 1_{s\times s} \end{bmatrix}.$$

$$SU(2k)^* = \{M \in SL(n,\mathbb{C}) : \Omega^{-1} M^{-\dagger} \Omega = M\}$$

$$\text{where } \Omega = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes 1_{k\times k} = \begin{bmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 0 & -1 & \\ & & 1 & 0 & \\ & & & & \ddots \end{bmatrix}.$$

The Orthogonal groups

$$SO(n,\mathbb{C}) = \{M \in SL(n,\mathbb{C}) : M^T 1 M = 1\}.$$

1 is a metric (non-degenerate symmetric form), normalized.

Real forms

$$SO(n,\mathbb{R}) = \{M \in SO(n,\mathbb{C}) : M^* = M\}$$

$$\cong \{M \in SL(n,\mathbb{R}) : M^T M = 1\} = \underline{SO(n)}.$$

$$SO(r,s) = \{M \in SO(n,\mathbb{C}) : g_{rs}^{-1} M^{-\dagger} g_{rs} = M\}$$

$$\cong \{M \in SL(n,\mathbb{R}) : g_{rs}^{-1} M^{-\dagger} g_{rs} = M\}$$

$M \mapsto \sqrt{g_{rs}}^{-1} M \sqrt{g_{rs}}$ is the isomorphism.

$$SO(2k)^* = \{M \in SO(n,\mathbb{C}) : \Omega^{-1} M^{-\dagger} \Omega = M\}.$$

The Symplectic groups

$$Sp(2n,\mathbb{C}) = \{M \in GL(2n,\mathbb{C}) : M^T \Omega M = \Omega\}.$$

Ω is a symplectic (non-degenerate antisymmetric) form, normalized.

Real forms

$$Sp(2n,\mathbb{R}) = \{M \in Sp(2n,\mathbb{C}) : M^* = M\}$$

$$Sp(2r,2s) = \{M \in Sp(2n,\mathbb{C}) : G_{rs}^{-1} M^{-\dagger} G_{rs} = M\}$$

where $G_{rs} = 1_{2\times 2} \otimes g_{rs}$.

$$Sp(2n) \equiv Sp(2n,0)$$

- There isn't a consensus across fields of practice for the names of the symplectic groups.

$Sp(2n,\mathbb{R})$ is also called $Sp(2n)$ or $Sp(n)$.

What we will call $Sp(2n)$ is also called $USp(2n)$ or $Sp(n)$.

The classification of simple finite-dimensional Lie groups.

Roughly speaking, there are only 5 basic classes of finite-dimensional Lie groups.

1) A_n: $SL(n+1,\mathbb{C})$ and its real forms
2) B_n: $SO(2n+1,\mathbb{C})$ " "
3) C_n: $Sp(2n,\mathbb{C})$ " "
4) D_n: $SO(2n,\mathbb{C})$ " "
5) 5 exceptions E_6, E_7, E_8, F_4, G_2 and their real forms (of which there are 4, 3, 2, 2, and 1, respectively.)