A supplement to the introduction to Lie groups and algebras by Chris Jackson — 26th of June, 2019

Week 1 Surveyed all finite simple groups and all simple finite-dimensional Matrix groups

Week 2
The derivatives of a matrix group at the identity form a Lie algebra.

Group multiplication $\xrightarrow{\text{1st order}}$ vector addition

Group multiplication $\xrightarrow{\text{2nd order}}$ Lie algebra multiplication

$$e^{-w\,dt}\,e^{-v\,dt}\,e^{w\,dt}\,e^{v\,dt} = 1 + [w, v]\,dt^2$$

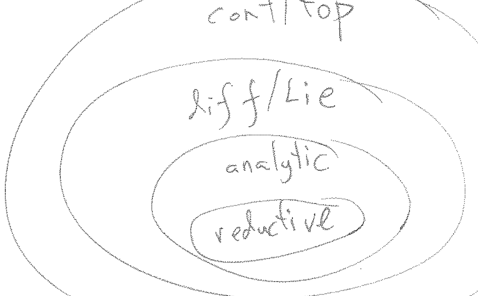
Thm
The local Lie algebra determines entirely the global group.

"analyticity," the spirit of exponentiation

$$f(a) = e^{a\frac{d}{dx}} f(0) \qquad e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \sum_{n=0}^{\infty}\frac{x^n}{n!} \text{ is the solution to}$$

$$\frac{dg}{dx} = g, \quad a = \int_0^a dx = \int_1^g g^{-1}\,dg.$$

~ emerges from continuity (closedness & path-connectedness)



Teaser S

① Hilbert's 5th problem

② $VW = \frac{[V,W]}{2} + \frac{\{V,W\}}{2}$

rigid dynamics — measurement theory

The adjoint representation

Lemma $e^X Y e^{-X} = Y + [X,Y] + \frac{1}{2}[X,[X,Y]] + \dots$

$= Y + ad_X(Y) + \frac{1}{2} ad_X^2(Y) + \dots$ where $ad_X(Y) = [X,Y]$

$= e^{ad_X}(Y).$

Pf $e^X Y e^{-X} = \lim_{n\to\infty} (e^{X/n})^n Y (e^{-X/n})^n$

$= \lim_{n\to\infty} \left(1 + \frac{1}{n} ad_X\right)^n (Y)$

$= e^{ad_X}(Y).$ □

Corollary: Let $Ad_g(X) = gXg^{-1}$. $Ad_{e^Y e^X} = e^{ad_Y} e^{ad_X}$.

Derivatives of ~~the exponential map~~

$$e^{X+\dot X dt} \neq e^{\dot X dt} e^X.$$

Rather, $e^{X+\dot X dt} = e^{f_X(\dot X)dt} e^X$ — i.e. $\frac{d}{dt} e^X = f_X(\dot X) e^X$.

Thm
$$f_X = \frac{e^{ad_X} - 1}{ad_X}.$$

Pf Consider $\Gamma(s,t) = \left(\frac{\partial}{\partial t}\Big|_s e^{sX(t)}\right) e^{-sX(t)}$.

$\frac{\partial}{\partial s}\Big|_t \Gamma = e^{sX} \dot X e^{-sX} = e^{s\,ad_X}(\dot X)$

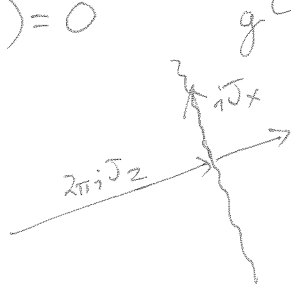
$f_X(\dot X) = \Gamma(1,t) = \Gamma(0,t) + \int_0^1 ds \frac{\partial \Gamma}{\partial s}$

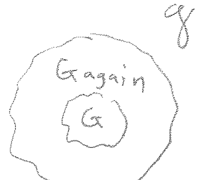
$= 0 + \frac{e^{ad_X}-1}{ad_X}(\dot X).$ □

E.g. $G = SU(2)$, $\mathfrak{g} = su(2)$

$$\frac{e^{ad_{2\pi i J_z}} - 1}{ad_{2\pi i J_z}}(iJ_z) = iJ_z$$

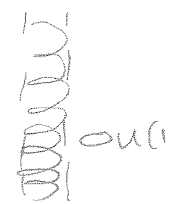
$$\frac{e^{ad_{2\pi i J_z}} - 1}{ad_{2\pi i J_z}}(iJ_x) = 0$$





just like

$ix \longmapsto e^{ix}$
$i\mathbb{R} \longmapsto u(1)$



The BCH formula

Similarly $e^{dX} e^{X} = e^{X + \varphi_X(dX)}$ where $\varphi_X = f_X^{-1} = \dfrac{\mathrm{ad}\,X}{e^{\mathrm{ad}\,X} - 1}$.

$\dfrac{x}{e^x - 1}$ is the generating function of the Bernoulli numbers.

Thm

$$e^{Y} e^{X} = \exp\left\{ X + \int_0^1 dt \, \frac{\log e^{t\,\mathrm{ad}\,Y} e^{\mathrm{ad}\,X}}{e^{t\,\mathrm{ad}\,Y} e^{\mathrm{ad}\,X} - 1} (Y) \right\}$$

Pf

Consider $e^{Z(t)} = e^{tY} e^{X}$.

$$\frac{d}{dt} e^{Z} = f_Z(\dot{Z}) e^{Z} = Y e^{Z} \implies Z(1) = Z(0) + \int_0^1 dt \, \dot{Z}$$

$$= X + \int_0^1 dt \, f_Z^{-1}(Y)$$

$$= X + \int_0^1 dt \, \frac{\log e^{\mathrm{ad}\,Z}}{e^{\mathrm{ad}\,Z} - 1}(Y)$$

$$= X + \int_0^1 dt \, \frac{\log e^{t\,\mathrm{ad}\,Y} e^{\mathrm{ad}\,X}}{e^{t\,\mathrm{ad}\,Y} e^{\mathrm{ad}\,X} - 1}(Y). \quad \square$$