An introduction to Lie groups and Lie algebras

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- What does it mean to be "continuous"?

This is an incredibly deep question.

The study of continuity is called Topology.

I think it is fair to say continuity in its modern notion can be broken into two notions we could call "closeness" and "connectedness."

"Closeness" is the topic of point-set topology.

"Connectedness" is the topic of algebraic topology.

- For the purpose of this course, we will not focus too much on continuity and it will suffice to define multivariate polynomials of $\mathbb{R}^{2n^2} \longrightarrow \mathbb{R}^2$ as continuous.

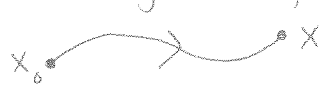
However, there are concepts that will prove very useful to guide the intuition:

~ A subset is <u>closed</u> if the limit of every converging sequence is an element of that subset.

E.g. $\mathbb{R}_0$, $e^{\mathbb{R}}$, and $GL(n,\mathbb{C})$ are not closed because zero is a limit.

$U(1)$, $SL(n,\mathbb{C})$, $SO(n,\mathbb{C})$, $S_p(n,\mathbb{C})$ and all their real forms are closed because they are the solutions to multivariate polynomial equations.

~ A subset is <u>path-connected</u> if between any two points there is a continuous path in the subset.



E.g. $SO(r,s)$ for $|r-s|>0$, $GL(n,\mathbb{R})$, $O(n,\mathbb{R})$, and $\mathbb{R}_0$ have two connected components.

All other simple Lie groups are path-connected.

- A group is <u>continuous</u> or <u>topological</u> if multiplication and inverses are continuous.
- A ~~ex~~ topological group is <u>Lie</u> if multiplication and inverses are differentiable.

~ Let G be a Lie group and define

$$\mathfrak{g} = \{\text{derivatives of } G \text{ at the identity.}\}$$

E.g. ~~The~~ If $G = e^{\mathbb{R}}$, then $\mathfrak{g} = \mathbb{R}$. Let $M = 1 + \epsilon X$

$$\mathfrak{u}(1) = i\mathbb{R}$$

$$\mathfrak{gl}(n,\mathbb{C}) = \{\text{any complex matrix}\}$$

$$\mathfrak{sl}(n,\mathbb{C}) = \{X \in \mathfrak{gl}(n,\mathbb{C}) : \operatorname{tr} X = 0\}$$

$$\mathfrak{so}(n,\mathbb{C}) = \{X \in \mathfrak{sl}(n,\mathbb{C}) : (\mathbb{1}X)^T = -(\mathbb{1}X)\}$$

$$\mathfrak{sp}(n,\mathbb{C}) = \{X \in \mathfrak{gl}(n,\mathbb{C}) : (\Omega X)^T = +(\Omega X)\}$$

$$\mathfrak{su}(r,s) = \{X \in \mathfrak{sl}(n,\mathbb{C}) : -g_{rs}^{-1} X^\dagger g_{rs} = X\}$$

$$= \left\{ \begin{bmatrix} A_{r\times r} & C_{s\times r} \\ B_{r\times s} & D_{s\times s} \end{bmatrix} \in \mathfrak{sl}(n,\mathbb{C}) : -A^\dagger = A,\ -D^\dagger = D,\ B^\dagger = C \right\}$$

- A Lie group is <u>analytic</u> if it is path-connected. if it is equal to its Taylor series. Recall that a function is analytic if it is equal to its Taylor series.

That means $f(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!} \left(\frac{d}{dx}\right)^n f(0) = e^{a\frac{d}{dx}}(f)(0)$.

Recall that ~~a~~ if a function over $\mathbb{C}$ is differentiable, then it is analytic.

<u>Hilbert's 5th Problem</u>

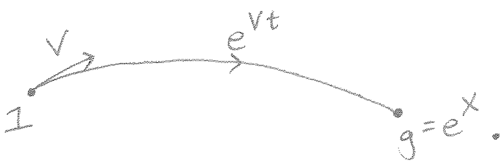
Are there topological groups that are not Lie groups?

Answer: Such a group would have "small subgroups."

• An analytic (Lie) group is <u>reductive</u> if it is a product with factors isomorphic to $U(1)$, $e^{\mathbb{R}}$, or any of the connected simple groups.

"Thm"

Let G be reductive. Then $g \in G \Longrightarrow g = e^X$ for some $X \in \mathfrak{g}$.



The Exponential

$$e^X = \lim_{n\to\infty}\left(1+\frac{X}{n}\right)^n = \sum_{n=0}^{\infty}\frac{X^n}{n!} \text{ is the solution to } \frac{dg}{dx} = g,\ x = \int_0^x dx = \int_1^g \frac{dg}{g}.$$

~ Let $V \in \mathfrak{g}$. $\{e^{Vt} : t \in \mathbb{R}\}$ is a 1-parameter subgroup, isomorphic to either $e^{\mathbb{R}}$ or $U(1)$.

"Thm"

The global multiplication rule of a Lie group is determined entirely by the local "structure" of $\mathfrak{g}$.

The "structure" of $\mathfrak{g}$

① $\mathfrak{g}$ is a ~~(real)~~ vector space over $\mathbb{R}$.

$$\boxed{\text{group multiplication} \overset{1^{st}\text{ order}}{\Longleftrightarrow} \text{vector addition}}$$

② Observe $e^{-Wdt}e^{-Vdt}e^{Wdt}e^{Vdt} = \left(1 - Wdt + \frac{W^2dt^2}{2}\right)\left(1 + e^{-Vdt}\cancel{dt}We^{Vdt} + \frac{W^2dt^2}{2}\right)$

$$= 1 - \cancel{Wdt} + \cancel{\frac{W^2dt^2}{2}} + \cancel{Wdt} - [V,W]dt^2 - \cancel{W^2dt^2} + \cancel{\frac{W^2dt^2}{2}} = 1 + [W,V]dt^2$$

where $[W,V] = WV - VW$.

$$\boxed{\text{group commutator} \overset{2^{nd}\text{ order}}{\Longleftrightarrow} \text{algebra multiplication}}$$

(2') $\mathfrak{g}$ is a Lie algebra over $\mathbb{R}$.

$$[X,Y] = -[Y,X]$$

$$[[X,Y],Z] = [X,[Y,Z]] - [Y,[X,Z]]$$

E.g. $\mathrm{tr}[X,Y] = \mathrm{tr}XY - \mathrm{tr}YX = 0$

$X^T = -X$ & $Y^T = -Y \Rightarrow [X,Y]^T = Y^TX^T - X^TY^T = -[X,Y]$

$(\Omega X)^T = \Omega X$ & $(\Omega Y)^T = \Omega Y \Rightarrow$

$$\Downarrow$$

$$\begin{aligned}
(\Omega[X,Y])^T &= (Y^TX^T\Omega^T - X^TY^T\Omega^T) \\
&= Y^T\Omega X - X^T\Omega Y \\
&= X^T\Omega^T Y - Y^T\Omega^T X \\
&= \Omega XY - \Omega YX \\
&= \Omega[X,Y]
\end{aligned}$$

Let $\{X_\mu\}$ be a basis of $\mathfrak{g}$.

$$[X_\mu, X_\nu] = f_{\mu\nu}^{\ \ \lambda} X_\lambda$$

these are called the structure constants of $\mathfrak{g}$.

The Baker-Campbell-Hausdorf Formula

- We shall now proceed to "prove" the "Thm" by explicitly calculating the BCH formula.
- Spiritually, what is significant here is that although it may seem ~~like~~ that in order to calculate $e^X e^Y$ one must define XY, it actually turns out one only needs to define $[X,Y] = XY - YX$. Since $XY = \frac{[X,Y]}{2} + \frac{\{X,Y\}}{2}$ where $\{X,Y\}$, one can say that incompressible motion within the group does not "experience" $\{X,Y\}$. The structure of $\{X,Y\}$ describes how the group has been embedded in a linear space. In a quantum theory, the $\{X,Y\}$ are equivalent to the spectra of measurement outcomes.
- Before directly diving into the BCH formula, there are two technical notions that are very important:

  1. The adjoint representation, and
  2. The derivatives (a.k.a. pushforwards and pullbacks) of the exponential.

The adjoint representation

• Lemma: $e^X Y e^{-X} = e^{ad_X}(Y)$ where $ad_X(Y) = [X,Y]$.

~~Pf~~ $e^X Y e^{-X} = \lim_{n\to\infty} (e^{X/n})^n \, Y \, (e^{-X/n})^n$

$$= \lim_{n\to\infty}\left(1+\frac{ad_X}{n}\right)^n (Y) = e^{ad_X}(Y).$$

Corollary: $e^Z = e^X e^Y \implies e^{ad_Z} = e^{ad_X} e^{ad_Y}$.

Derivatives of the exponential

$e^{X+\dot{X}dt} \neq e^{\dot{X}dt} e^X$. Rather, $e^{X+\dot{X}dt} = e^{f_X(\dot{X})dt} e^X$

or equivalently, $\frac{d}{dt} e^X = f_X(\dot{X}) e^X$.

• Thm: $f_X = \dfrac{e^{ad_X}-1}{ad_X} = \sum_{n=0}^{\infty} \dfrac{ad_X^n}{(n+1)!}$.

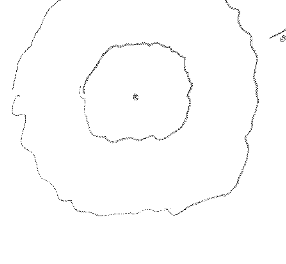
~~Pf~~ Consider $\Gamma(s,t) = \frac{\partial}{\partial t}\left(e^{sX(t)}\right)\Big|_s e^{-sX(t)}$.

$$\left.\frac{\partial \Gamma}{\partial s}\right|_t = \frac{\partial}{\partial t}\left(e^{sX}X\right)e^{-sX} - \frac{\partial}{\partial t}\left(e^{sX}\right)Xe^{-sX} = e^{sX}\dot{X}e^{-sX} = e^{s\,ad_X}(\dot{X}).$$

$$\frac{d}{dt}(e^X)e^{-X} = \Gamma(1,t) = \Gamma(0,t) + \int_0^1 ds\,\frac{\partial\Gamma}{\partial s} = \frac{e^{ad_X}-1}{ad_X}(\dot{X}).$$

E.g. Let $X = 2\pi i J_z$. $f_{\cancel{X}}(J_z) = J_z$ & $f_X(J_x) = f_X(J_y) = 0$


$2\pi i J_z$

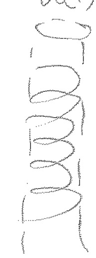

su(2)

infinitely covers SU(2).

Similar to $x \longmapsto e^{ix}$

$\mathbb{R} \longrightarrow u(1)$

u(1)



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• Similarly, $e^{dX} e^{X} = e^{X + \varphi_X(dX)}$ where $\varphi_X = f_X^{-1} = \dfrac{ad_X}{e^{ad_X}-1}$.

φ_X is the generating function of the Bernoulli numbers

$$\varphi_X = \sum_{n=0}^{\infty} \frac{B_n}{n!} ad_X^n .$$

BCH formula

Thm: $$\log e^Y e^X = X + \int_0^1 dt \, \frac{\log(e^{tY} e^{X})}{e^{t\,ad_Y} e^{ad_X} - 1} (Y).$$

~~Pf~~

Consider $e^{Z(t)} = e^{tY} e^{X}$.

$$f_Z(\dot{Z}) e^{Z} = \frac{d}{dt} e^{Z} = Y e^{Z}.$$

$$\therefore \log e^Y e^X = Z(1) = Z(0) + \int_0^1 dt \, \varphi_Z(Y)$$

$$= X + \int_0^1 dt \, \frac{ad_Z}{e^{ad_Z} - 1} (Y)$$

$$= X + \int_0^1 dt \, \frac{\log(e^{t\,ad_Y} e^{ad_X})}{e^{t\,ad_Y} e^{ad_X} - 1} (Y). \blacksquare$$

Spiritual Summary

We have demonstrated that the multiplication rule of an analytic Lie group is determined entirely by the Lie algebra of its derivatives at the identity.