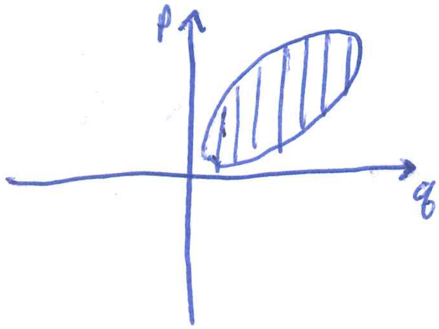
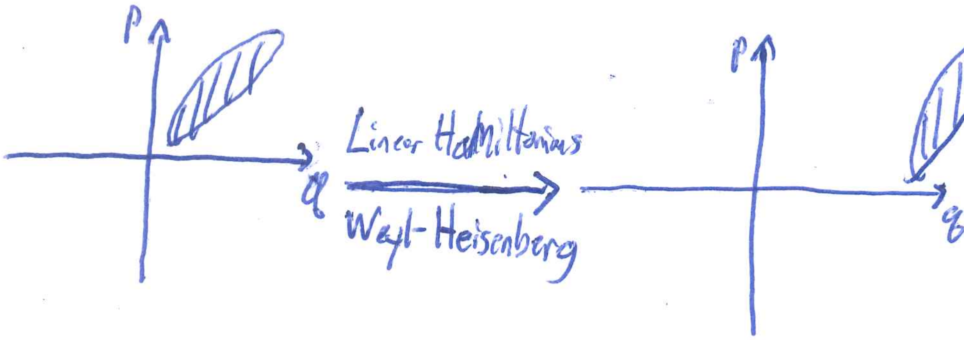
Non-Compact Semi-Simple Lie Groups

The state of a quantum particle or a bosonic field may be represented by a ~~pr~~ quasi-probability distribution on phase space

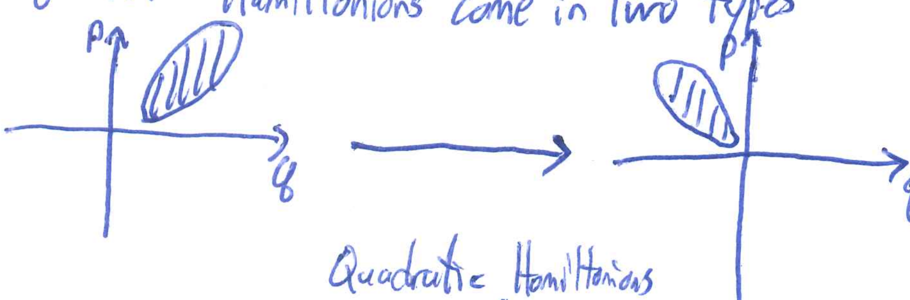


We will be studying two types of transformations which can be applied to these states. ~~Transformations~~ Translations



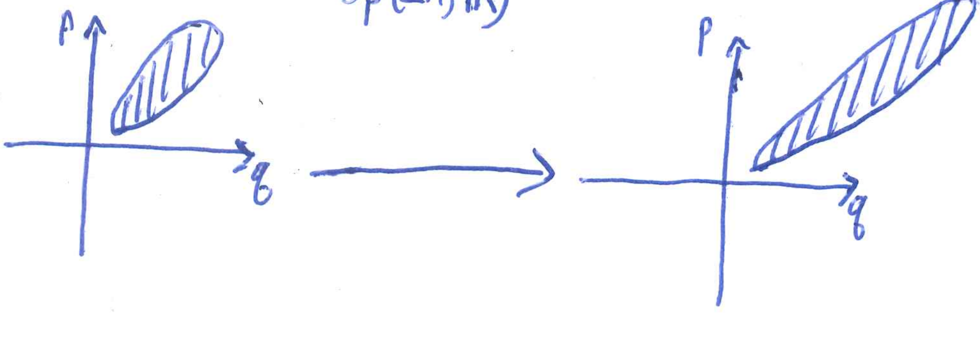
which we will see are generated by linear Hamiltonians. The associated unitaries will be representations of the Weyl-Heisenberg group. The transformations generated by quadratic Hamiltonians come in two types

Rotations:



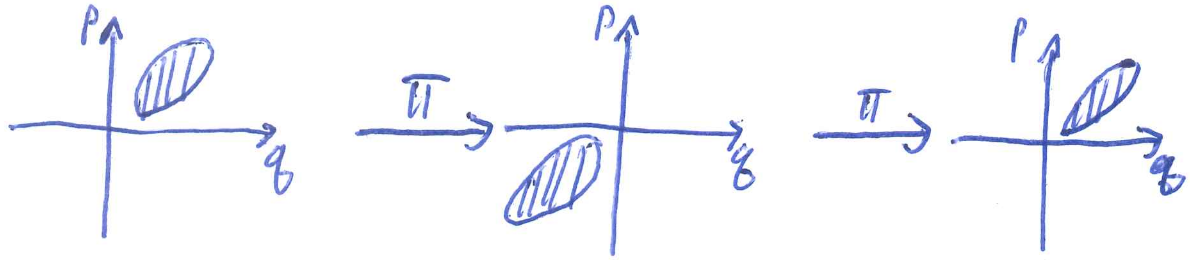
Quadratic Hamiltonians
$Sp(2n, \mathbb{R})$

Squeezing:



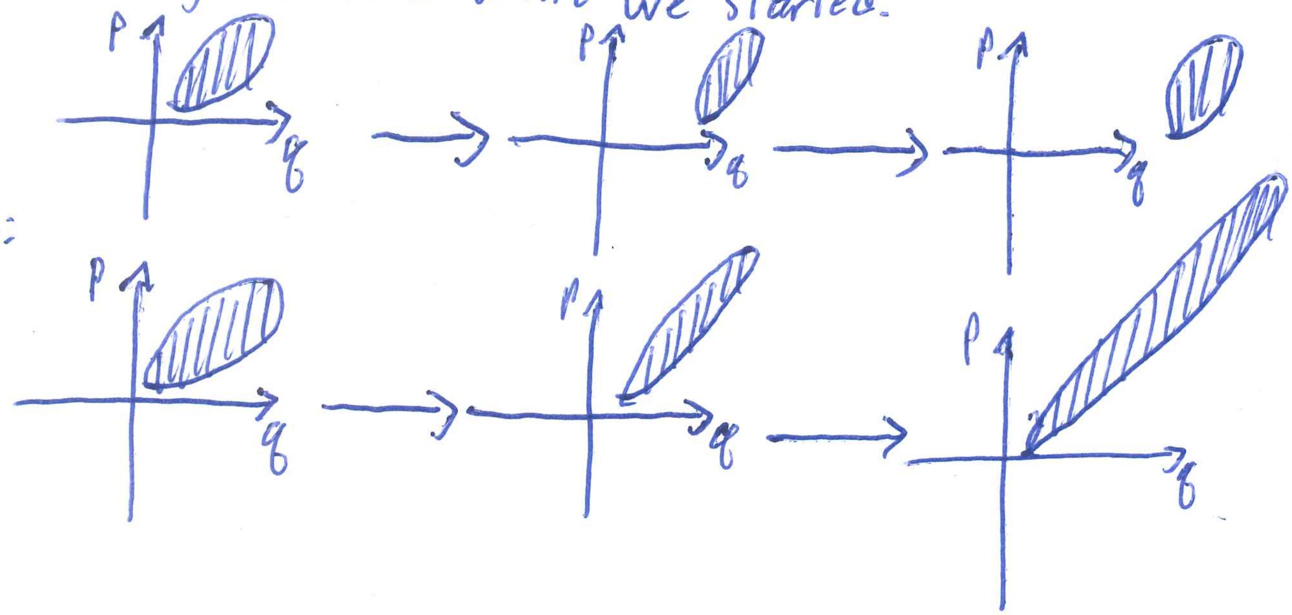
This time the unitaries will be a representation of $Sp(2n, \mathbb{R})$. There is a qualitative difference between rotations and the other two types of transformations. A rotation by 2π can be identified with the identity

Compact:



On the other hand, we may translate or squeeze as much as we like and we will never get back to where we started.

Non-Compact:



As a final preliminary comment, we note that all of these operations preserve the volume in phase space.

I. The Weyl-Heisenberg Group

In the early days of quantum mechanics, Born, Heisenberg, and Jordan discovered the canonical commutation relations

$$[Q_j, P_k] = i\hbar \delta_{jk} \mathbb{1}$$

Soon after, Weyl realized that these objects formed a representation of a Lie algebra defined by

$$[X, Y] = Z$$
$$[X, Z] = [Y, Z] = 0$$

At this point, we could proceed to study the Weyl-Heisenberg group in the abstract but it will be simpler to work with more concrete objects. To this end, we need a definition.

Definition: A representation (π, V) of a group G is a homomorphism

$$\pi : g \in G \rightarrow \pi(g) \in GL(V)$$

where $GL(V)$ is the group of invertible maps $V \rightarrow V$, with V a vector space.

In the context of group representations, homomorphism means

$$\pi(g_1)\pi(g_2) = \pi(g_1 g_2)$$

A representation of a Lie algebra is defined in an analogous way except now

$$[\pi(g_1), \pi(g_2)] = \pi([g_1, g_2])$$

One representation of the Weyl-Heisenberg algebra is given by

$$X \longrightarrow A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Y \longrightarrow B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Z \longrightarrow I = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

An easy way to verify that these matrices satisfy the defining relations of the Lie algebra is to note that the only non-zero product ~~of these~~ among them is

$$AB = I$$

This is partially due to the fact that these matrices nilpotent, i.e. they satisfy

$$M^n = 0$$

for some n, in this case n=2. We can determine the group elements by exponentiating these matrices

$$e^{aA+bB+cI} = \mathbb{1}_3 + aA + bB + cI + \tfrac{1}{2}abAB = \mathbb{1}_3 + aA + bB + (c+\tfrac{1}{2}ab)I$$

$$= \begin{pmatrix} 1 & a & c+\frac{1}{2}ab \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

So the group representation we have found consists of 3x3 upper triangular matrices of determinant one.

Another important representation, sometimes called the Schrödinger representation, is given by the position and momentum operators in the quantum mechanics of particles or by the quadrature operators in theories of bosonic fields.

$$X \longrightarrow -iQ$$

$$Y \longrightarrow -iP$$

$$Z \longrightarrow -i\mathbb{1}$$

which satisfy

$$[-iQ, -iP] = -i\mathbb{1}$$

in units where $\hbar = 1$. The generalization to more degrees of freedom is

$$[-iQ_j, -iP_k] = -i\delta_{jk}\mathbb{1}$$

Now lets compute the group commutator. This will be easier in the 3x3 representation since we have the relation

$$\cancel{e^{-bB} = \mathbb{1}_3 + bB} \qquad e^{aA} = \mathbb{1}_3 + aA$$

The commutator then is

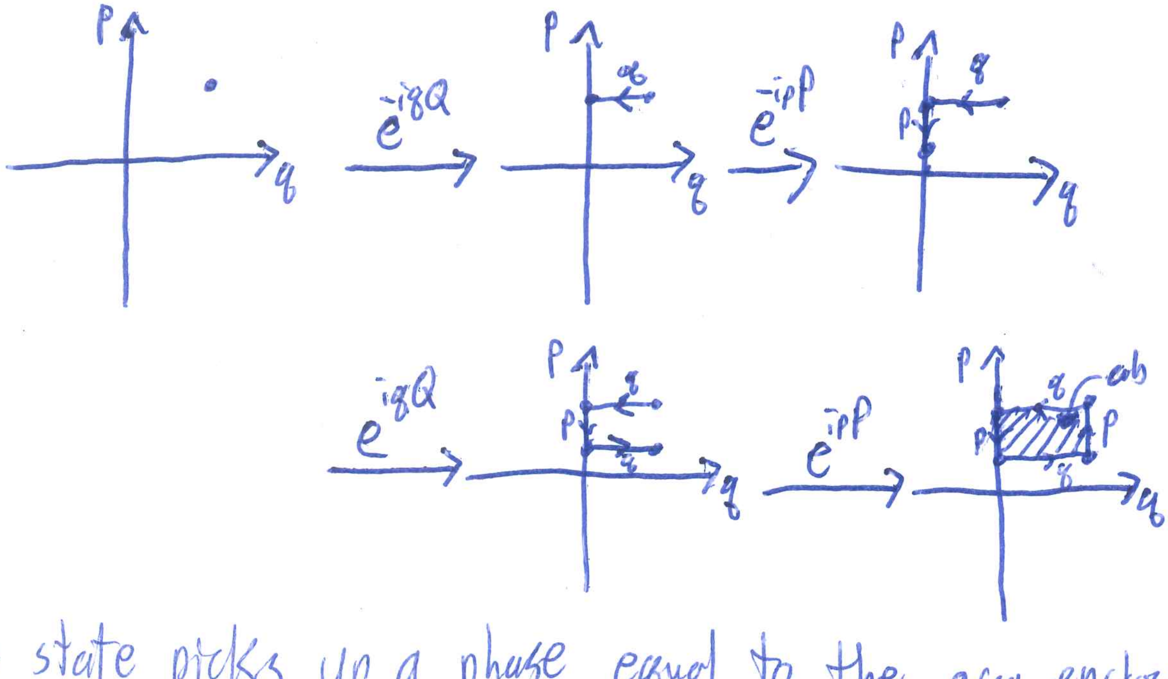
$$e^{-bB}e^{-aA}e^{bB}e^{aA} = e^{-bB}e^{be^{-aA}Be^{aA}} = e^{-bB}e^{bB-ab[A,B]} = e^{-bB}e^{bB-abI}$$

$$= e^{-bB}e^{bB}e^{-abI} = e^{-abI}$$

where the fourth equality from $[B, I] = 0$. In the Schrödinger representation, the result reads

$$e^{ipP}e^{iqQ}e^{-ipP}e^{-iqQ} = e^{iab\mathbb{1}}$$

which has a simple geometric interpretation on phase space



So the state picks up a phase equal to the area enclosed in phase space. Before moving on to quadratic Hamiltonians, we should

State two important theorems without proof.

Theorem: The Schrödinger representation is irreducible.

Theorem: (Stone–von Neumann) Any irreducible representation of the ~~group~~ Weyl–Heisenberg group/algebra satisfying

$$Z \to -i\mathbb{1}$$

is unitarily equivalent to the Schrödinger representation

II. $Sp(2n, \mathbb{R})$

A symplectic form is a matrix

$$\Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

The Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$ is defined to be $\{A \mid \Omega A + A^T \Omega = 0\}$.

Theorem: Any matrix A that can be written as

$$A = \Omega M \qquad M = M^T$$

is in $\mathfrak{sp}(2n, \mathbb{R})$.

Proof: First, note that

$$\Omega^2 = -\mathbb{1}$$

and

$$\Omega^T = -\Omega$$

Then we have

$$\Omega A + A^T \Omega = \Omega \Omega M + (\Omega M)^T \Omega = -\mathbb{1} M + M^T \Omega^T \Omega = -M - M \Omega \Omega$$

$$= -M + M\mathbb{1} = -M + M = 0$$

Actually, this theorem goes both directions. To see this take A to be in $sp(2n,\mathbb{R})$ and write it as

$$A = \Omega M$$

for some (not assumed to be symmetric) M. We then have

$$\Omega(\Omega M) + M^T \Omega^T \Omega = -M - M^T(\Omega\Omega) = -M + M^T = 0$$

which is exactly the statement that M is symmetric $M = M^T$.

We are going to show that algebra is isomorphic to the set of quadratic Hamiltonians. To this end, we're going to introduce some notation. First, if we have n degrees of freedom

$$\vec{X} \equiv \begin{pmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_n \\ P_1 \\ P_2 \\ \vdots \\ P_n \end{pmatrix}$$

It will also be convient to have

$$[\vec{X}_1, \vec{X}_2] \equiv \vec{X}_1 \vec{X}_2^T - (\vec{X}_2 \vec{X}_1^T)^T$$

The canonical commutation relations then read

$$[-i\vec{X}, -i\vec{X}] = -i\Omega$$

An arbitrary linear Hamiltonian can be written as

$$\vec{a} \cdot \vec{X}$$

with $\vec{a} \in \mathbb{R}^n$. Quadratic Hamiltonians can be written as

$$\vec{x}^T M \vec{x}$$

with $M = M^T$. It will be convenient to introduce some factors of $-i$ and $\frac{1}{2}$. The next step is to work out the commutators.

Lemma. [Quadratic, Linear] = Linear or $\left[-\frac{i}{2}\vec{x}^T M\vec{x}, -i\vec{a}\cdot\vec{x}\right] = -i\vec{a}'\cdot\vec{x}$

Proof: In order to prove this, we will use the fact that in this representation elements of the Lie algebra (quadratic Hamiltonians) can be viewed as products of operators so that we have the important formula

$$[(AB), C] = A[B, C] + [A, C]B$$

We then have, in the summation convention

$$\left[-\frac{i}{2}\vec{x}^T M\vec{x}, -i\vec{a}\cdot\vec{x}\right] = -\frac{1}{2}\left[x_j M_{jk} x_k, a_l x_l\right] = -\frac{M_{jk}a_l}{2}\left[x_j x_k, x_l\right] = -\frac{M_{jk}a_l}{2}\Big(x_j[x_k, x_l] + [x_j, x_l]x_k$$

$$= -i\frac{M_{jk}a_l}{2}\left(x_j\Omega_{kl} + x_k\Omega_{jl}\right) = -iM_{jk}\Omega_{kl}a_l x_j$$

$$= -i(M\Omega\vec{a})\cdot\vec{x}$$

So we have found that $\vec{a}' = M\Omega\vec{a}$.

Lemma. $\left[-\frac{i}{2}\vec{x}^T M\vec{x}, -\frac{i}{2}\vec{x}^T N\vec{x}\right] = -\frac{i}{2}\vec{x}^T(M\Omega N - N\Omega M)\vec{x}$

Proof: $\left[-\frac{i}{2}\vec{x}^T M\vec{x}, -\frac{i}{2}\vec{x}^T N\vec{x}\right] = -\frac{1}{4}M_{jk}N_{lm}\left[x_j x_k, x_l x_m\right] = -\frac{M_{jk}N_{lm}}{4}\Big(x_l[x_j x_k, x_m] + [x_j x_k, x_l]x_m$

$$= -\frac{M_{jk}N_{lm}}{4}\Big(x_l x_j[x_k, x_m] + x_l[x_j, x_m]x_k + x_j[x_k, x_l]x_m + [x_j, x_l]x_k x_m$$

$$= -i\frac{M_{jk}N_{lm}}{4}\left(x_l x_j\Omega_{km} + x_l x_k\Omega_{jm} + x_j x_m\Omega_{kl} + x_k x_m\Omega_{jl}\right)$$

$$= -i\frac{M_{jk}N_{lm}}{2}\left(x_l x_j \Omega_{km} + x_j x_m \Omega_{kl}\right) = -\frac{i}{2}\vec{x}^T(M\Omega N - N\Omega M)\vec{x}$$

where the fifth equality follows from the symmetry of M and N.

Theorem: Quadratic Hamiltonians are isomorphic to $sp(2n,\mathbb{R})$.

Proof: The isomorphism is given by

$$\frac{-i}{2}\vec{x}^T M\vec{x} \longleftrightarrow \Omega M$$

We have already established that ΩM is in $sp(2n,\mathbb{R})$ so we only need to show that ΩM and ΩN have the correct commutator

$$[\Omega M, \Omega N] = \Omega M\Omega N - \Omega N\Omega M = \Omega(M\Omega N - N\Omega M)$$

We need to be careful here. Just because the quadratic Hamiltonians are isomorphic to $sp(2n,\mathbb{R})$ does not mean $\exp\left[-\frac{i}{2}\vec{x}^T M\vec{x}\right]$ gives us $Sp(2n,\mathbb{R})$. In fact, we get a double cover of $Sp(2n,\mathbb{R})$ called the metaplectic group. However, unitaries generated in this way differ from elements of a representation of ~~the~~ $Sp(2n,\mathbb{R})$ by at most a phase so expectation values will not be effected but Schrödinger picture states may pick up an overall phase.

In order to learn more about what quadratic Hamiltonians do, lets examine the Heisenberg evolution of the linear operators

$$\dot{\vec{x}} = \frac{i}{2}[\vec{x}^T M\vec{x}, \vec{x}] = -\frac{1}{2}M\Omega\vec{x}$$

which can be solved to read

$$\vec{x} = e^{-\frac{1}{2}M\Omega t}\,\vec{x}$$

where $\exp[-\frac{1}{2}M\Omega t]$ is in $Sp(2n,\mathbb{R})$.

Theorem. Quadratic Hamiltonians preserve commutation relations.

Proof: If U is the unitary generated by a quadratic Hamiltonian we have

$$[\vec{x},\vec{x}] \to [U^\dagger \vec{x} U, U^\dagger \vec{x} U] = [S\vec{x}, S\vec{x}]$$

where S is the symplectic matrix corresponding to U. We can evaluate this commutator

$$[S\vec{x},S\vec{x}] = S\vec{x}\vec{x}^T S^T - (S\vec{x}\vec{x}^T S^T)^T = S\left(\vec{x}\vec{x}^T - (\vec{x}\vec{x}^T)^T\right)S^T$$

$$= iS\Omega S^T = i\Omega$$

Another way of saying this is that the symplectic form is preserved. What does this mean? Consider the product

$$\vec{v}_1^T \Omega_2 \vec{v}_2 = (x_1\ y_1)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = x_1 y_2 - y_1 x_2$$

which is equal to the area of the parallelogram with $\vec{v}_1$ and $\vec{v}_2$ as sides. So preserving the symplectic form means preserving areas; in this case areas in phase space.

In order to gain some intuition, lets look at what happens if we have only one degree of freedom. In this case, any polynomial can be written as a linear combination of three polynomials

$$K_x = -\frac{1}{2}\vec{x}^T Z \vec{x}$$

$$K_y = -\tfrac{i}{2}\vec{x}^T \mathbb{1}_2 \vec{x}$$

$$K_z = -\tfrac{i}{2}\vec{x}^T X \vec{x}$$

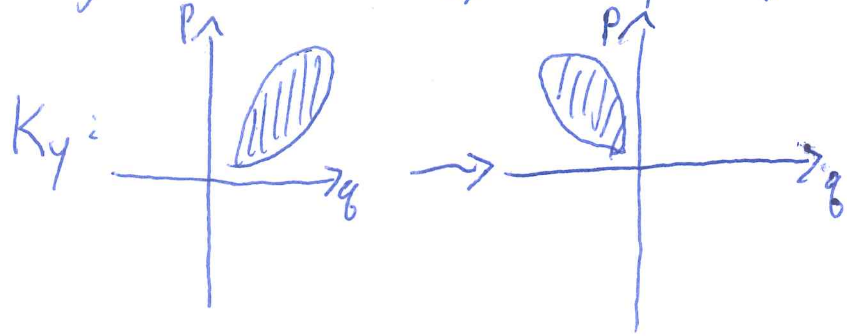
Under the isomorphism discussed above we have

$$K_x \longleftrightarrow \Omega_2 Z = -X$$

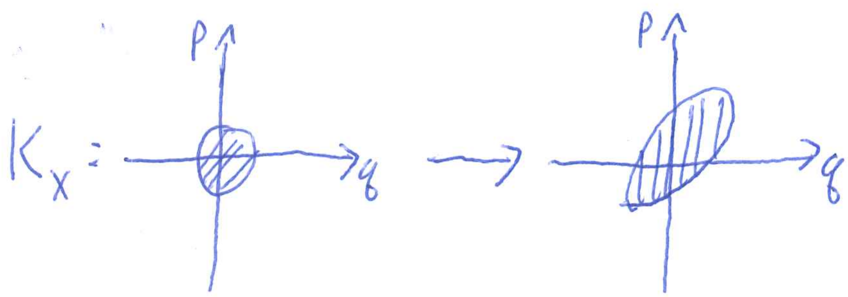
$$K_y \overset{?}{\longleftrightarrow} \Omega_2 \mathbb{1}_2 = iY$$

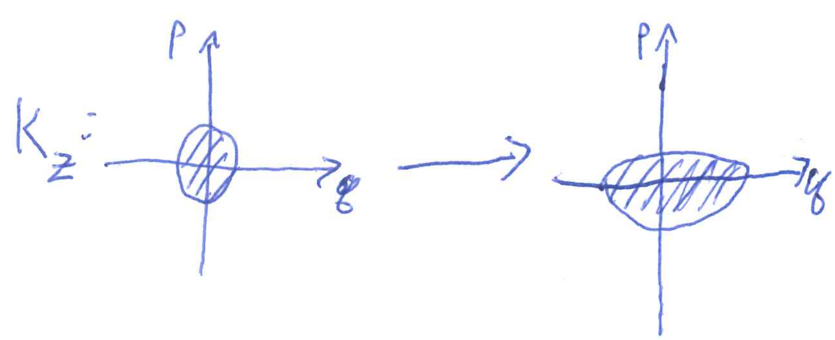
$$K_z \longleftrightarrow \Omega_2 X = Z$$

It is a coincidence that the object isomorphic to K_y has an i in front of it. This generator gives us rotations, the compact part of the group.



The other two give us squeezing.





There is no unitary which transforms the compact generator, K_y, into the noncompact generators, K_x, K_z, but the noncompact generators are transformed into each by unitaries generated by the compact generator. This provides some intuition for our final topic.

III. Bloch-Messiah Decomposition

From the pictures we've drawn so far, it seems intuitive that the effect of a unitary generated by a quadratic Hamiltonian can be broken down into a rotation in phase space, a squeezing operation, and a second rotation (linear optics, squeezing, linear optics). This is made precise by the KAK decomposition of this group known as the Bloch-Messiah decomposition.

The relevant Cartan involution is

$$K_j \to e^{-i\frac{\pi}{2}K_y} K_j e^{i\frac{\pi}{2}K_y}$$

The effect of which is

$$K_x \to e^{-i\frac{\pi}{2}K_y} K_x e^{i\frac{\pi}{2}K_y} = -K_x$$

$$K_y \to e^{-i\frac{\pi}{2}K_y} K_y e^{i\frac{\pi}{2}K_y} = K_y$$

$$K_z \to e^{-i\frac{\pi}{2}K_y} K_z e^{i\frac{\pi}{2}K_y} = -K_z$$

In the language of KAK decompositions this means $K_y \in K$ and $K_x, K_z \in P$. This is exactly the decomposition we discussed intuitively. In the case of more degrees of freedom, the fact that the middle matrix is diagonal implies that we need only single mode squeezers. The two outer matrices must be orthogonal. To see this note that

$$\vec{x}^T \vec{x} = \vec{x}^T \mathbb{1} \vec{x} = |\vec{x}|^2$$

is the magnitude of $\vec{x}$ in phase space. Physically, this is essentially the energy. So that just as symplectic matrices had to obey

$$S \Omega S^T = \Omega$$

to preserve area, operations that preserve energy (like linear optics) must obey

$$O \mathbb{1} O^T = O O^T = \mathbb{1}$$