Exercises for session 1 (June 5th)

Exercises taken (with modifications) from B. Hall - Lie Groups, Lie Algebras and Representations: An Elementary Introduction. Springer – Graduate Texts in Mathematics

Exercise 1. Consider the general linear matrix group $GL(n, \mathbb{C})$. A *matrix Lie group* is any closed subgroup G of $GL(n, \mathbb{C})$. The closure relation means that, given a sequence A_m of matrices in G, if it converges to some invertible matrix A, then $A \in G$. In this exercise we construct an example of a subgroup of $GL(n, \mathbb{C})$ which is not closed, and thus it is not a matrix Lie group.

Given an irrational number $a \in \mathbb{R}$, consider the set

$$G = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{ita} \end{pmatrix}, t \in \mathbb{R} \right\}$$

a. Prove that G is a subgroup of $GL(2, \mathbb{C})$.

b. If $A = -I$, prove that $A \notin G$ (I is the identity matrix).

c. Show that G is not a matrix Lie group. For this, prove that it is possible to create a sequence of elements of G which converges to $-I$. Hint: use the fact that $e^{i\pi ma}$ is dense in the unit circle, .

Exercise 2. Show that

a. $A \in O(n)$ if and only if A preserves the bilinear form $B(x, y) \equiv \langle x, y \rangle_{\mathbb{R}} = \sum_{i=1}^{n} x_i y_i$, i.e. $B(Ax, Ay) = B(x, y)$ for all vectors $x, y \in \mathbb{R}^n$.

b. $A \in U(n)$ if and only if A preserves the bilinear form $B(x, y) \equiv \langle x, y \rangle_{\mathbb{C}} = \sum_{i=1}^{n} x_i^* y_i$, i.e. $B(Ax, Ay) = B(x, y)$ for all vectors $x, y \in \mathbb{C}^n$.

c. $A \in Sp(2n, \mathbb{R})$ if and only if A preserves the bilinear form $B(x, y) = \sum_{i=1}^{n} (x_i y_{n+i} - x_{n+i} y_i)$, i.e. $B(Ax, Ay) = B(x, y)$ for all vectors $x, y \in \mathbb{R}^{2n}$.

Exercise 3. Following Exercise 2, for n, k positive integers consider $\mathbb{R}^{n+k}$ and define the bilinear form

$$B(x, y) = \sum_{i=1}^{n} x_i y_i - \sum_{i=n+1}^{n+k} x_i y_i$$

We can then define the **generalized orthogonal group** $O(n; k)$ as the set of matrices A in $GL(n + k, \mathbb{R})$ such that $B(Ax, Ay) = B(x, y)$ for all vectors $x, y \in \mathbb{R}^{n+k}$.

a. Let $g = \begin{pmatrix} I_n & 0 \\ 0 & -I_k \end{pmatrix}$. Show that for all $x, y \in \mathbb{R}^{n+k}$, $B(x,y) = \langle x, gy \rangle_{\mathbb{R}}$.

b. Show that $A \in O(n;k)$ if and only if $A^T g A = g$

c. Show that the matrix

$$A(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

is in $SO(1;1)$ and check that $A(t_1)A(t_2) = A(t_1 + t_2)$.

d. For comparison, show that the matrix

$$B(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

is in $SO(2)$ and check that $B(\phi_1)B(\phi_2) = B(\phi_1 + \phi_2)$.

Exercise 4. Show that every element of $A \in SU(2)$ can be written as

$$\begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}$$

with $\alpha, \beta \in \mathbb{C}$ satisfying $|\alpha|^2 + |\beta|^2 = 1$. In turn this condition implies that $SU(2)$ can be viewed as a three-dimensional sphere sitting on $\mathbb{C}^2 = \mathbb{R}^4$.