Exercises for session 2 (June 12th)

Exercises taken (with modifications) from B. Hall - Lie Groups, Lie Algebras and Representations: An Elementary Introduction. Springer – Graduate Texts in Mathematics

Exercise 1. Directly verify the following theorem for the case of $SU(n)$

Theorem: Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. Let X, Y be elements of $\mathfrak{g}$ and A an element of G. Then

a. $AXA^{-1} \in \mathfrak{g}$

b. $sX \in \mathfrak{g}$, *for all real numbers* s

c. $X + Y \in \mathfrak{g}$

d. $[X, Y] \in \mathfrak{g}$

Exercise 2: *Adjoint mapping.* Let G be a matrix Lie group, with Lie algebra $\mathfrak{g}$. Then, for each $A \in G$, define a linear map $\mathrm{Ad}_A : \mathfrak{g} \to \mathfrak{g}$ by the formula $\mathrm{Ad}_A(X) = AXA^{-1}$. This also implies the existence of an associated real linear map $\mathrm{ad}_X(Y) = [X, Y]$.

a. Show by induction that $(\mathrm{ad}_X)^m(Y) = \sum_{k=0}^{m} \binom{m}{k} X^k Y (-X)^{m-k}$. Alternatively, just convince yourself of the result by direct calculation up to $m = 3$.

b. Prove that $e^{\mathrm{ad}_X}(Y) = e^X Y e^{-X} = \mathrm{Ad}_{e^x}(Y)$

c. Show that $\mathrm{ad}_X([Y, Z]) = [\mathrm{ad}_X(Y), Z] + [Y, \mathrm{ad}_X(Z)]$

Exercise 3: *$SU(2)$ and $SO(3)$*. In this (longer) exercise we will explore the connection between the groups $SU(2)$ and $SO(3)$ and the corresponding Lie algebras $su(2)$ and $so(3)$. In particular we will see that the groups are connected by a *homomorphism*, while the algebras are actually *isomorphic*.

Part A: Lie algebra homomorphisms. Let $\mathfrak{g}$ and $\mathfrak{h}$ be matrix Lie algebras. A linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ is a *Lie algebra homomorphism* if $\phi([X, Y]) = [\phi(X), \phi(Y)]$ for all $X, Y \in \mathfrak{g}$. If, in addition, ϕ is also one-to-one, then ϕ is called a *Lie algebra isomorphism*.

a. Write down the most general form of the elements of $so(3)$ and $su(2)$ and from them derive a basis for each of the spaces.

b. Show that $su(2)$ and $so(3)$ are isomorphic. In order to do that, construct the injective map ϕ that connects both spaces (since they are linear, we can define ϕ by its action on a basis).

Part B: Lie group homomorphisms. Let G and H be matrix Lie groups. A *Lie group homomorphism* is a map $\Phi : G \to H$ such that $\Phi(g_1 g_2) = \Phi(g_1)\Phi(g_2)$ for all $g_1, g_2 \in G$[1]. If, in addition, Φ is also one-to-one[2], then Φ is called a *Lie group isomorphism.*

Let $G = SU(2)$ and consider the space $\mathbb{V}$ of all 2×2 complex matrices which are hermitian and have zero trace. Of course, the Pauli matrices form a basis of $\mathbb{V}$:

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

With respect to the Hilbert-Schmidt inner product $\langle A, B \rangle = \frac{1}{2}\mathrm{Tr}(AB)$, the set $\{A_1, A_2, A_3\}$ is orthonormal. Since $\mathbb{V}$ is a real vector space, we can then identify it with $\mathbb{R}^3$:

$$(x_1, x_2, x_3) \to x_1 A_1 + x_2 A_2 + x_3 A_3 \tag{2}$$

a. Let $U \in SU(2)$ and $A \in \mathbb{V}$. Show that $UAU^{-1} \in \mathbb{V}$.

b. Consider the linear map $\Phi_U(A) : \mathbb{V} \to \mathbb{V}$, $\Phi_U(A) = UAU^{-1}$.

   (a) Show that $\Phi_{U_1 U_2} = \Phi_{U_1}\Phi_{U_2}$.

   (b) Show that Φ_U is an element of $O(3)$. Recalling Ex. 2. a. from Session 1, this can proven by showing that Φ_U leaves invariant the inner product defined above (i.e. that Φ_U is an orthogonal transformation of $\mathbb{V}$.)

   (c) From this conclude that $\Phi : SU(2) \to O(3)$ is a Lie group homomorphism.

c. Remembering Ex. 4 from Session 1, construct the actual map $\Phi(\alpha, \beta)$ that takes an element $U(\alpha, \beta)$ of $SU(2)$ to an orthogonal 3×3 matrix. We can do this by acting on the basis elements of $\mathbb{V}$ with a generic U

d. The *kernel* of a group homomorphism $\Phi : G \to H$ is defined as

$$\ker(\Phi) = \{g \in G : \ \Phi(g) = e_H\}, \tag{3}$$

where e_H is the identity element of H. Show that $\ker(\Phi) = \{-\mathbb{I}, \mathbb{I}\} \simeq \mathbb{Z}_2$. Note that since the kernel is not trivial, then Φ is not a Lie group isomorphism.

[1]... and Φ is continuous
[2]... and Φ^{-1} is continuous