Ex 1

$$G = \left\{ \overbrace{\begin{pmatrix} e^{it} & 0 \\ 0 & e^{ita} \end{pmatrix}}^{g_t}, t \in \mathbb{R} \right\} \quad a \text{ irrational}$$

a) $G \subset GL(2, \mathbb{C})$ i) $g_t g_s = \begin{pmatrix} e^{i(t+s)} & 0 \\ 0 & e^{i(t+s)a} \end{pmatrix} = g_{t+s} \in G$ ✓

· subgroup: ii) $g_{t=0} = \mathbb{I} \in G$ ✓

iii) $\forall g_t$, I can define $g_{-t} \mid g_t g_{-t} = \mathbb{I}$ ✓

b) $-\mathbb{I} \notin G \Rightarrow e^{it} = -1 \Leftrightarrow t = (2n+1)\pi \quad n \geq 0$

but $e^{ita} = e^{ia(2n+1)\pi} \neq -1$

c) $e^{iam\pi}$ is dense in the circle $\Rightarrow$ meaning that I can make $e^{iam\pi}$ arbitrarily close to $e^{i\phi}$ $(\phi \in [0, 2\pi))$ by properly choosing m.

$\Rightarrow$ there is a sequence $A_m \subset G$ that converges to $-\mathbb{I} \notin G \Rightarrow G$ is not a matrix Lie Group

Ex 2

(a) $O(n) = \{ A \in GL(n,\mathbb{R}) \mid A^T A = I \}$

$B(x,y) = \sum_k x_k y_k \qquad (Az)_i = \sum_j A_{ij} z_j$

$$\Rightarrow B(Ax, Ay) = \sum_i \sum_{j\ell} A_{ij} x_j A_{i\ell} y_\ell$$

$$= \sum_{j\ell} x_j y_\ell \sum_i A_{ij} A_{i\ell} \qquad A_{ij} = (A^T)_{ji}$$

$$= \sum_{j\ell} x_j y_\ell \sum_i (A^T)_{ji} A_{i\ell}$$

$$= \sum_{j\ell} x_j y_\ell \underbrace{(A^T A)_{j\ell}}_{= \delta_{j\ell}} = \sum_j x_j y_j \checkmark$$

$A \in O(n) \Rightarrow$

(b) $U(n) = \{ A \in GL(n,\mathbb{C}) \mid A^\dagger A = I \}$

$B(x,y) = \sum_k x_k^* y_k$

$$B(Ax, Ay) = \sum_i \sum_{j\ell} (A_{ij} x_j)^* A_{i\ell} y_\ell$$

$$= \sum_{j\ell} x_j^* y_\ell \sum_i A_{ij}^* A_{i\ell} \qquad A_{ij}^* = (A^\dagger)_{ji}$$

$$= \sum_{j\ell} x_j^* y_\ell \sum_i (A^\dagger)_{ji} A_{i\ell}$$

$$= \sum_{j\ell} x_j^* y_\ell \underbrace{(A^\dagger A)_{j\ell}}_{\delta_{j\ell}} = \sum_j x_j^* y_j \checkmark$$

$A \in U(n)$

(c) $Sp(2n, \mathbb{R}) = \{ A \in GL(2n, \mathbb{R}) \mid A^T J A = J \}$ $\quad J = \begin{pmatrix} 0 & +I \\ -I & 0 \end{pmatrix}$

$$B(x,y) = \sum_{k=1}^{n} x_k y_{n+k} - x_{n+k} y_k \qquad (Az)_i = \sum_j A_{ij} z_j$$

obs: $B(x,y) = \langle x, Jy \rangle_{\mathbb{R}}$ because $J_{ij} = \delta_{j,n+i} - \delta_{i,n+j}$

$$= \sum_{k=1}^{2n} x_k \sum_{l} J_{kl} y_l$$

$$= \sum_{k,l} x_k y_l \left(\delta_{l,n+k} - \delta_{k,n+l} \right) = \sum_k x_k y_{n+k} - x_{n+k} y_k$$

So, $B(Ax, Ay) = \langle Ax, JAy \rangle_{\mathbb{R}}$

$$= \sum_i (Ax)_i (JAy)_i$$

$$= \sum_i \sum_{jl} A_{ij} x_j (JA)_{il} y_l$$

$$= \sum_{jl} x_j y_l \sum_i A_{ij} (JA)_{il} \qquad A_{ij} = (A^T)_{ji}$$

$$= \sum_{jl} x_j y_l \underbrace{\sum_i (A^T)_{ji} (JA)_{il}}_{(A^T J A)_{jl} \underset{A \in Sp(2n;\mathbb{R})}{=} J_{jl}}$$

$$= \sum_{jl} x_j J_{jl} y_l = \sum_j x_j (Jy)_j$$

$$= \langle x, Jy \rangle_{\mathbb{R}} = B(x,y) \checkmark$$

Ex 3.

$$B(x,y) = \sum_{i=1}^{n} x_i y_i - \sum_{i=n+1}^{n+k} x_i y_i$$

$$O(n;k) = \{ A \in GL(n+k;\mathbb{R}) \mid B(Ax,Ay) = B(x,y) \ \forall x,y \}$$

a) $g_{ij} = \begin{cases} \delta_{ij} & i,j \leq n \\ -\delta_{ij} & n+1 \leq i,j \leq n+k \end{cases} \Rightarrow (gy)_i = \sum_j g_{ij} y_j = \begin{cases} y_i & i \leq n \\ -y_i & i > n \end{cases}$

$$\downarrow$$

$$B(x,y) = \sum_{i=1}^{n+k} x_i \left(\sum_{j=1}^{n+k} g_{ij} y_j \right)$$

$$= \langle x, gy \rangle_{\mathbb{R}} \checkmark$$

b) Idem ex 2 c

c) $A(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \in O(1;1)$

$g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad A^T = A \quad \Big\} \ A^T g = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cosh t & -\sinh t \\ \sinh t & -\cosh t \end{pmatrix}$

$$A^t g A = \begin{pmatrix} \cosh t & -\sinh t \\ \sinh t & -\cosh t \end{pmatrix} \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

$$= \begin{pmatrix} \cosh^2 t - \sinh^2 t & 0 \\ 0 & \sinh^2 t - \cosh^2 t \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = g \checkmark$$

Ex 4

$$SU(2) = \{ A \in GL(2;\mathbb{C}) \mid A^{\dagger}A = \mathbb{I} \wedge \det A = 1 \}$$

$$A = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \quad \text{with } \alpha, \beta, \gamma, \delta \in \mathbb{C}$$

$$A^{\dagger} = \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix} \Rightarrow A^{\dagger}A = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}\begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix} = \begin{pmatrix} |\alpha|^2 + |\gamma|^2 & \alpha\beta^* + \gamma\delta^* \\ \beta\alpha^* + \delta\gamma^* & |\beta|^2 + |\delta|^2 \end{pmatrix}$$

$$\parallel$$

$$AA^{\dagger} = \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix}\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} = \begin{pmatrix} |\alpha|^2 + |\beta|^2 & \alpha^*\gamma + \beta^*\delta \\ \gamma^*\alpha + \delta^*\beta & |\gamma|^2 + |\delta|^2 \end{pmatrix}$$

So, from $AA^{\dagger}$! $\rightarrow \delta = \underbrace{-\frac{\gamma}{\beta^*}}_{x}\alpha^*$ and $\gamma = \underbrace{-\frac{\delta}{\alpha^*}}_{y}\beta^*$

and $y = -\frac{\delta}{\alpha^*} = -\frac{x\alpha^*}{\alpha^*} \Rightarrow y = -x$

So, $A = \begin{pmatrix} \alpha & y\beta^* \\ \beta & x\alpha^* \end{pmatrix} = \begin{pmatrix} \alpha & -x\beta^* \\ \beta & x\alpha^* \end{pmatrix}$

And $\det(A) = 1 = x \underbrace{(|\alpha|^2 + |\beta|^2)}_{\text{1 from eqn for } AA^{\dagger} = \mathbb{I}} \Rightarrow x = 1$ $\rightarrow$ $A = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}$ ✓

with $|\alpha|^2 + |\beta|^2 = 1$