SESSION 2

Ex 1

$$su(n) = \{ X \in gl(n,\mathbb{C}) \mid \overbrace{X^+ = -X}^{i)} \wedge \overbrace{\operatorname{tr} X = 0}^{ii)} \} = g$$

$$SU(n) = \{ A \in GL(n,\mathbb{C}) \mid A^+ A = 1 \wedge \det A = 1 \} = G$$

a) $\tilde{X} = AXA^{-1}$

$$\left.\begin{array}{l} i)\ \tilde{X}^+ = (A^+)^{-1} X^+ A^+ = -AXA^{-1} = -\tilde{X} \checkmark \\ ii)\ \operatorname{tr}\tilde{X} = \operatorname{tr}(A^{-1}AX) = 0 \checkmark \end{array}\right\} \tilde{X} \in g$$

b) $\tilde{X} = sX \quad (s \in \mathbb{R})$

$$\left.\begin{array}{l} i)\ \tilde{X}^+ = sX^+ = -sX = -\tilde{X} \checkmark \\ ii)\ \operatorname{tr}\tilde{X} = s\operatorname{tr}X = 0 \checkmark \end{array}\right\} \tilde{X} \in g$$

c) $\tilde{X} = X + Y$

$$\left.\begin{array}{l} \to i)\ \tilde{X}^+ = X^+ + Y^+ = -(X+Y) = -\tilde{X} \checkmark \\ \hookrightarrow ii)\ \operatorname{tr}\tilde{X} = \operatorname{tr}X + \operatorname{tr}Y = 0 \checkmark \end{array}\right\} \tilde{X} \in g$$

d) $\tilde{X} = [X, Y]$

$$\left.\begin{array}{l} i)\ \tilde{X}^+ = (XY - YX)^+ = Y^+X^+ - X^+Y^+ = YX - XY = -\tilde{X} \\ ii)\ \operatorname{tr}\tilde{X} = \operatorname{tr}XY - \operatorname{tr}YX = 0 \checkmark \end{array}\right\} \tilde{X} \in g$$

Ex 2

a) $\mathrm{ad}_X Y = [X,Y]$

$$(\mathrm{ad}_X)^2 Y = [X,[X,Y]]$$
$$= [X, XY - YX]$$
$$= [X,XY] - [X,YX]$$
$$= XXY - 2XYX + YXX$$

$$(\mathrm{ad}_X)^3 Y = [X,[X,[X,Y]]]$$
$$= X^3Y - 2X^2YX + XYX^2 - X^2YX + 2XYX^2 - YX^3$$
$$= X^3Y - 3X^2YX + 3XYX^2 - YX^3$$

$$(\mathrm{ad}_X)^m Y = \sum_{k=0}^{m} \binom{m}{k} X^k Y (-X)^{m-k} \qquad \binom{m}{k} = \frac{m!}{k!(m-k)!}$$

$$\mathrm{ad}_X Y = \binom{1}{0} Y(-X) + \binom{1}{1} XY = XY - YX \checkmark$$

$$\mathrm{ad}_X^2 Y = \binom{2}{0} Y(-X)^2 + \binom{2}{1} XY(-X) + \binom{2}{2} X^2Y$$
$$= +YX^2 - 2XYX + X^2Y \checkmark$$

$$\mathrm{ad}_X^3 Y = \binom{3}{0} Y(-X)^3 + \binom{3}{1} XY(-X)^2 + \binom{3}{2} X^2Y(-X) + \binom{3}{3} X^3Y$$
$$= -YX^3 + 3XYX^2 - 3X^2YX + X^3Y \checkmark$$

b) $e^{\mathrm{ad}_X}(Y) = e^X Y e^{-X}$

$$\downarrow$$

$$\sum_m \frac{(\mathrm{ad}_X)^m (Y)}{m!} = \sum_m \sum_{k=0}^{m} \underbrace{\frac{1}{m!}\binom{m}{k}}_{\frac{1}{\cancel{m!}} \frac{\cancel{m!}}{k!(m-k)!}} X^k Y (-X)^{m-k}$$

$$= \sum_m \sum_{k=0}^{m} \frac{1}{k!} X^k Y \frac{1}{(m-k)!} (-X)^{m-k}$$

$$\boxed{\sum_{m=0}^{\infty} \sum_{k=0}^{m} \xrightarrow{l = m-k} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty}}$$

$$= \left(\sum_{k=0}^{\infty} \frac{X^k}{k!}\right) Y \left(\sum_{l=0}^{\infty} \frac{(-X)^l}{l!}\right)$$

$$= e^X Y e^{-X} = \mathrm{Ad}_{e^X} Y$$

c) $\mathrm{ad}_X([Y,Z]) = [X,[Y,Z]]$

$$= [X, YZ - ZY]$$

$$= [X,YZ] - [X,ZY]$$

$$= Y[X,Z] + [X,Y]Z - [X,Z]Y - Z[X,Y]$$

$$= \underbrace{Y[X,Z] - [X,Z]Y} + \underbrace{[X,Y]Z - Z[X,Y]}$$

$$= [Y, \mathrm{ad}_X(Z)] + [\mathrm{ad}_X(Y), Z]$$

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Ex 3

Part A: $su(2) = \{X \in sl(2,\mathbb{C}) : X^{\dagger} = -X\}$

$so(3) = \{Y \in sl(3,\mathbb{R}) : Y^{T} = -Y\}$

a)

$$X \in su(2) \Rightarrow X = i\begin{pmatrix} a_3 & a_1 + i a_2 \\ a_1 - i a_2 & -a_3 \end{pmatrix}$$

$$i\begin{pmatrix} 0 & +i \\ -i & 0 \end{pmatrix} \qquad = a_1\overbrace{(i\sigma_x)}^{X_1} + a_2\overbrace{(-i\sigma_y)}^{X_2} + a_3\overbrace{(i\sigma_z)}^{X_3}$$

$$Y \in so(3) \Rightarrow Y = \begin{pmatrix} 0 & b_3 & b_2 \\ -b_3 & 0 & b_1 \\ -b_2 & -b_1 & 0 \end{pmatrix} = b_1\underbrace{\begin{pmatrix} 0&0&0 \\ 0&0&1 \\ 0&-1&0 \end{pmatrix}}_{Y_1} + b_2\underbrace{\begin{pmatrix} 0&0&1 \\ 0&0&0 \\ -1&0&0 \end{pmatrix}}_{Y_2} + b_3\underbrace{\begin{pmatrix} 0&1&0 \\ -1&0&0 \\ 0&0&0 \end{pmatrix}}_{Y_3}$$

obs:
- $[Y_1, Y_2] = Y_3 \quad [Y_2, Y_3] = Y_1 \quad [Y_3, Y_1] = Y_2$ $\rightarrow [Y_i, Y_j] = \varepsilon_{ijk} Y_k$
- $[X_1, X_2] = -i^2[\sigma_x, \sigma_y] = 2i\sigma_z = 2X_3$
- $[X_2, X_3] = -i^2[\sigma_y, \sigma_z] = 2i\sigma_x = 2X_1$ ✓
- $[X_3, X_1] = i^2[\sigma_z, \sigma_x] = 2(-i\sigma_y) = 2X_2$ $\rightarrow [X_i, X_j] = 2\varepsilon_{ijk} X_k$

b) $\phi : su(2) \rightarrow so(3)$ s.t. $\phi(X_k) = 2 \cdot Y_k \quad (k = 1,2,3)$

and of course $\phi^{-1}(Y_k) = \frac{1}{2} X_k$

$$\Rightarrow \phi([X_i, X_j]) \overset{?}{=} [\phi(X_i), \phi(X_j)]$$
$$\phi(2\varepsilon_{ijk} X_k) \overset{?}{=} 4[Y_i, Y_j]$$
$$4\varepsilon_{ijk} Y_k \overset{?}{=} 4\varepsilon_{ijk} Y_k$$
$$Y_k = Y_k \checkmark$$

$\boxed{\text{So, } \phi \text{ is a Lie Algebra isomorphism between } su(2) \text{ and } so(3)}$

Part B

a) $U \in SU(2)$, $A \in \mathbb{V}$ with $\mathbb{V} = \{$ 2x2 complex matrices, hermitian and zero trace $\}$

$$\tilde{A} = UAU^{-1}$$

$$\left.\begin{array}{l} \cdot\ tr(\tilde{A}) = tr\, U^{-1}UA = tr\, A = 0 \ \checkmark \\ \cdot\ \tilde{A}^{\dagger} = (UAU^{-1})^{\dagger} = (U^{-1})^{\dagger} A^{\dagger} U^{\dagger} = UAU^{-1} \ \checkmark \end{array}\right\} \tilde{A} \in \mathbb{V}$$

b) i) $\Phi_U(A) : \mathbb{V} \to \mathbb{V}$, $\Phi_U(A) = UAU^{-1}$

$$\Phi_{U_1U_2}(A) = U_1U_2A(U_1U_2)^{-1}$$
$$= U_1U_2AU_2^{-1}U_1^{-1} = U_1\, \Phi_{U_2}(A)\, U_1^{-1}$$
$$= \Phi_{U_1}\left(\Phi_{U_2}(A)\right) \ \checkmark$$

ii) $\langle \Phi_U(A), \Phi_U(B) \rangle = \frac{1}{2} tr\left(UAU^{-1}UBU^{-1}\right)$

$$= \frac{1}{2} tr\left(UABU^{-1}\right) = \frac{1}{2} tr(AB) = \langle A, B \rangle \ \checkmark$$

$$\Rightarrow \Phi_U \in O(3)$$

c) $U(\alpha, \beta) = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}$

$$\cdot\ UA_1U^{-1} = Re(\alpha^2 - \beta^2)A_1 - Im(\alpha^2 + \beta^2)A_2 - 2Re(\alpha\beta)$$

$$U A_2 U^{-1} = -\operatorname{Im}(-\alpha^2+\beta^2)\, A_1 + \operatorname{Re}(\alpha^2+\beta^2)\, A_2 - 2\operatorname{Im}(\alpha\beta)\, A_3$$

$$U A_3 U^{-1} = 2\operatorname{Re}(\beta\alpha^*)\, A_1 - 2\operatorname{Im}(\alpha\beta^*)\, A_2 + (|\alpha|^2 - |\beta|^2)\, A_3$$

So, $$\Phi(\alpha,\beta) = \begin{pmatrix} \operatorname{Re}(\alpha^2-\beta^2) & +\operatorname{Im}(\beta^2-\alpha^2) & 2\operatorname{Re}(\alpha^*\beta) \\ -\operatorname{Im}(\alpha^2+\beta^2) & \operatorname{Re}(\alpha^2+\beta^2) & -2\operatorname{Im}(\alpha\beta^*) \\ -2\operatorname{Re}(\alpha\beta) & -2\operatorname{Im}(\alpha\beta) & |\alpha|^2-|\beta|^2 \end{pmatrix}$$