Topological Quantum Computing
Homework 3

July 4, 2006

1. **The Fundamental Group I.** In this problem, we will construct a mathematical object known as the *fundamental group*, or *first homotopy group* of a connected space R. The fundamental group is a topological invariant of the space R, meaning that two spaces with distinct fundamental groups are necessarily topologically distinct (the converse is not true). We begin by introducing the notion of homotopy.

A *homotopy* is a continuous function $H(t,x)$ parameterized by $t \in [0,1]$ between two continuous functions $f(x)$ and $g(x)$ such that $H(0,x) = f(x)$ and $H(1,x) = g(x)$. The two functions f and g connected by the homotopy H are said to be *homotopic*. From this definition, we see that a homotopy is just a continuous transformation between two continuous functions that both map from some domain space into some other range space.

Homotopy is a *transitive* property. Suppose that f is homotopic to g and g is homotopic to h. Then there exists a homotopy H_1 between f and g and a homotopy H_2 between g and h. Intuitively, if f and g are connected by a smooth deformation, and g and h are too, then surely f and h are also connected by a smooth deformation. With the notion of homotopy, we can show this rigorously. By composing the maps H_1 and H_2 and re-parameterizing t, we can construct a homotopy H_3 between f and h. Indeed,

$$H_3(t,x) = \begin{cases} H_1(2t,x) & 0 \le t \le 1/2 \\ H_2(2t-1,x) & 1/2 \le t \le 1 \end{cases} \tag{1}$$

is one such choice, which explicitly shows that f and h are homotopic. Furthermore, homotopies may always be "inverted" by letting $t \to 1-t$ so that $H(0,x) = g(x)$ and $H(1,x) = f(x)$. Thus homotopy is also *symmetric*. Since every function is obviously homotopic to itself, homotopy is also *reflexive*. Therefore homotopy is an *equivalence relation*, and the set of all functions on a space partition into disjoint *equivalence classes* of homotopic functions. We write $f \sim g$ to denote that two functions f and g are homotopic and denote the equivalence class of a function f by $[f]$ and say that f is a *representative* of the class $[f]$.

When we get to the fundamental group, it will be these equivalence classes which are the objects of interest. The functions we will consider will be closed directed loops within the space R. More specifically, the functions will map from the unit interval into the space R, with the requirement that for every function f, there exists some point $x_0 \in R$ with $f(0) = f(1) = x_0$. The point x_0 is some arbitrary point in the space R that may be distinct for different functions. Note that what we are calling "loop" is a little more general than what is normally implied by "curve", since for example the function $f(x) = x_0$ is a valid loop, as are any other manner of curves that self-intersect, retrace themselves or stop at some point for some subinterval of $[0,1]$. The only thing we require of our loops is that they be continuous and start and end at x_0.

There is also a restricted notion of homotopy, known as based homotopy. A *based homotopy at* x_0 introduces the additional restriction that the homotopy map remains "fixed" at some base point x_0, which is the *same* for *all* functions, while t moves from 0 to 1. That is, we introduce the additional requirement that $H(t,0) = H(t,1) = x_0$ for all t and fixed (but arbitrary) x_0. If a based homotopy at x_0 exists between two functions f and g, we say that they are *homotopic at* x_0 and write $f \sim g$, since it will be clear from context whether this equivalence relation is the based or unbased version. When we wish to emphasize that we are talking about a regular rather than based homotopy, we speak of *free homotopy* and say that f and g are *freely homotopic*.

We now define a product of two loops at x_0, exactly as we composed two homotopies above. We first traverse the loop f and then the loop g. More formally,

$$f \circ g(x) = \begin{cases} f(2x) & 0 \le x \le 1/2 \\ g(2x-1) & 1/2 \le x \le 1 \ . \end{cases} \tag{2}$$

The product law Eq. 2 for the product of two loops is unsuitable for the definition of a group. In particular, it fails to be associative due to parameterization issues, and, more seriously, the product of two loops necessarily passes through the point x_0 at $x = 1/2$, which puts stringent limitations on how or whether an arbitrary loop can be decomposed as a product of other loops. Thus, we will try to use this with equivalence classes of loops.

a) Argue that the product of two classes is naturally defined by

$$[f] \circ [g] = [f \circ g] \ , \tag{3}$$

and that this definition makes sense.

When talking about equivalence classes of loops, we use Greek letters, primarily α and β.

b) Show that the set of all equivalence classes of loops at x_0 on R with the product given by Eq. 3 forms a group. This group is called the *fundamental group at* x_0, and is denoted $\pi_1(R, x_0)$.

The subscript 1 is there because there are other groups, the so-called *higher homotopy groups* which are associated not with maps of the unit interval into R, but with S_n, the surface of the unit sphere in Euclidean $n+1$ space, into R.

Now we introduce a natural generalization of a loop: a path. For a *path* in R we relax the requirement on a loop that it start and end at the same point. If f is a path in R, and $f(0) = x_0$ and $f(1) = x_1$, then we say that f is a path connecting x_0 to x_1. Paths may be composed just as loops were if the final point of the first path is the initial point of the second path.

c) Consider the based fundamental groups $\pi_1(R, x_0)$ and $\pi_1(R, x_1)$. By considering a path p connecting x_0 to x_1, show that the groups $\pi_1(R, x_0)$ and $\pi_1(R, x_1)$ are isomorphic (this is known as a *path isomorphism*). Since this holds for arbitrary base points, we speak of *the* fundamental group of a space R, and denote it simply by $\pi_1(R)$.

d) Show that the path isomorphism is unique if and only if the group $\pi_1(R)$ is Abelian. Here unique means that the path isomorphism between equivalence classes is totally independent of the choice of path, though there will be many choices of paths for which we can construct isomorphisms. (Hint 1: two isomorphic images of $\pi_1(R, x_1)$ differ by an inner automorphism (i.e. an overall conjugation) of $\pi_1(R, x_0)$. Hint 2: If p is a path from x_0 to x_1, and q is a path connecting x_1 to x_0, then $p \circ q$ is a *loop* at x_0.)

e) Two loops at x_0 which are freely homotopic are not-necessarily homotopic at x_0. Show that two loops at x_0 are freely homotopic if and only if they are in the same conjugacy class of $\pi_1(R, x_0)$. Compare this result to the result we saw in class about non-Abelian anyons that said that fluxons belonging to the same conjugacy class were indistinguishable, though they may come in many types.

Finally, here is an additional fact about the fundamental group: Suppose the space R is a *product space*, i.e. R can be written as the topological product $R_1 \times R_2$. For example, the torus is the product $S_1 \times S_1$; it decomposes as two copies of the unit circle. Similarly, a cylinder is the product of a line with the unit circle, and $\mathbb{R}^n$ is the product of n copies of the real line. Then the fundamental group satisfies

$$\pi_1(R) = \pi_1(R_1) \times \pi_1(R_2),$$

where the product on the right is the group direct product.