Topological Quantum Computing
Homework 4

July 11, 2006

1. **Anyons in an A_4 Gauge Theory.** The group A_4 is the group of even permutations of four letters. It has the geometric interpretation as the group of proper rotations of a tetrahedron. In this problem, we consider an anyon model with A_4 symmetry.

a) Calculate the possible fluxes and calculate the centralizer subgroup for each of the fluxes.

b) Calculate the character table for each of the centralizer subgroups. You will find it convenient to introduce the notation $\omega = e^{2\pi i/3}$. What charges are possible for each flux?

You found in parts *a* and *b* that there are 14 particles in an A_4 anyon model. Label these particles p_0 through p_{13}, where p_0 is the vacuum.

c) What is the dimension of each of the particles? Use this to verify that the total dimension of the theory is equal to 12, the order of A_4.

d) By calculating the class algebra for A_4, determine which flux fusions (ignoring charge) are consistent with conservation of magnetic flux.

e) By calculating the multiplication table for the representation ring of A_4, derive the fusion rules for the pure charge sector of the theory.

f) Calculate the modular S-matrix. This is a lengthy calculation, but don't worry, it is straightforward and many of the elements are zero. Check that $S^* = S^{-1} = S^3$.

g) By calculating the modular T-matrix, find the topological spin for each of the particles in the A_4 spectrum.

h) Use Verlinde's formula to check your answers from part *e*. For one pair of particles, the fusion algebra has a curious factor of 2 in it. What does this mean?

i) Describe the braid matrix $\mathcal{R}$ for the pure charge sector. Don't write it out; it's 36×36. Use this to check the pure charge part of your answers from part *f*.

j) Check that the generalized spin-statistics theorem holds for the pure charge sector.

2. **Braiding in an S_4 Anyon Model.** The smallest group that has a particle with both a flux and a charge each with dimension > 1 is S_4, the symmetric group on four letters. This group has five conjugacy classes given by the class representatives $\{(\ ), (1\ 2), (1\ 2)(3\ 4), (1\ 2\ 3), (1\ 2\ 3\ 4)\}$. The third group element, $(1\ 2)(3\ 4)$, has a total of three elements in its conjugacy class. The size of the centralizer of this group element is $24/3 = 8$. A minimal generating set for this centralizer is $\{(1\ 3)(2\ 4), (1\ 4\ 2\ 3)\}$, and you can convince yourself that this is isomorphic to the dihedral group D_4 of symmetries of the square. A transversal for this centralizer is given by the coset representatives $\{(\ ), (2\ 3), (2\ 4)\}$. This ordering for the transversal generates the elements of the conjugacy class in the following order: $\{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$.

Since D_4 (the centralizer) is non-Abelian, it must have at least one irrep with dimension > 1. Indeed, the faithful irrep given by rotations and reflections of a square in a plane is two dimensional. Thus this particle has multi-dimentional flux and charge. We will investigate the braiding properties of this particle, which I'll call η.

a) The braid matrix $\mathcal{R}$ is block diagonal in the tensor product space of all particles. Write down the braid matrix for the space $\eta \otimes \eta$. This is a 36×36 matrix, but it's not so bad.

b) What is the modular S-matrix element $S_{\eta\eta}$?