

# Topological Quantum Computing Solution 1

June 22, 2006

## 1. Spin-Statistics for Abelian Anyonic “Molecules”.

a) The charge  $q_j$  in the molecule on the right passes half-way around each of the fluxes  $\Phi_k$ . This is true for all  $j$ . The molecule on the left has a similar phase, so the total phase is

$$\exp\left(i\sum_{j,k}q_j\Phi_k\right)=\exp\left(i\sum_jq_j\sum_k\Phi_k\right)=e^{iQ\Theta},$$

where  $Q$  and  $\Theta$  are the total charge and total flux in the molecule, respectively.

b) Each charge  $q_j$  passes around each of the fluxes  $\Phi_k$ , and this is true for all  $j$ . Again, we pick up a total phase of  $e^{iQ\Theta}$ .

## 2. A Hamiltonian for Abelian Anyons.

a) Since  $\alpha(\mathbf{q}_j-\mathbf{q}_k)=\alpha(\mathbf{q}_k-\mathbf{q}_j)+\pi$ ,  $\dot{\alpha}$  is symmetric under interchange. We can then symmetrize the sum, picking up a factor of two since we are doubling the number of terms. Note that  $\dot{\alpha}(\mathbf{q}_j-\mathbf{q}_j)=\dot{\alpha}(\mathbf{0})=0$ , since however we choose to define  $\alpha(\mathbf{0})$  it is always a constant.

b) The action  $S$  is given by

$$S=\int_0^T dt\mathcal{L}=S_0+\int_0^T dtV_{\text{topo}}$$

where  $S_0$  is the action for the unperturbed Lagrangian  $L_0$ . Since  $V_{\text{topo}}$  has a total time derivative, we can integrate immediately to obtain

$$S=S_0-\frac{\theta\hbar}{\pi}\sum_{j<k}(\alpha(\mathbf{q}_j-\mathbf{q}_k)|_{t=T}-\alpha(\mathbf{q}_j-\mathbf{q}_k)|_{t=0}).$$

The extra phase in the amplitude  $e^{iS/\hbar}$  that comes from  $V_{\text{topo}}$  is  $-\theta\Delta\alpha$ , where

$$\Delta\alpha=\frac{1}{\pi}\sum_{j<k}(\alpha(\mathbf{q}_j-\mathbf{q}_k)|_{t=T}-\alpha(\mathbf{q}_j-\mathbf{q}_k)|_{t=0})$$

c) From part b, the topological phase is just  $e^{-i\theta\Delta\alpha}$ , so let’s calculate  $\Delta\alpha$ . We are free to choose our coordinates so that particles 1 and 2 lie on the x-axis. Let’s introduce the convenient notation  $\alpha_{jk}(T):=\alpha(\mathbf{q}_j-\mathbf{q}_k)|_{t=T}$ . Then  $\alpha_{12}(0)=0$  and  $\alpha_{12}(T)=\pi$ , so  $\Delta\alpha=-1$ , and the total topological phase is  $e^{i\theta}$ , as expected. If we exchanged the anyons clockwise instead, we obtain  $\alpha_{12}(T)=-\pi$ , and the topological phase is  $e^{-i\theta}$ , as expected. Although the final points are the same in both cases, the “approach” to the final point has a different limit in each case. This goes back to the fact that arctan is a multi-valued function.

Now suppose we add a third particle. We don’t know what  $\alpha_{13}$  or  $\alpha_{23}$  are, but we will see that they cancel.

$$\Delta\alpha=-1+\frac{1}{\pi}(\alpha_{13}(T)-\alpha_{13}(0))+\frac{1}{\pi}(\alpha_{23}(T)-\alpha_{23}(0)).$$

But the initial angle that particle 1 makes with particle 3 is the same as the final angle that particle 2 makes with particle 3, since they swap positions. So  $\alpha_{13}(0)=\alpha_{23}(T)$ . Similarly,  $\alpha_{23}(0)=\alpha_{13}(T)$ .

From the above expression for  $\Delta\alpha$ , we see that everything cancels except the term we had for the case  $n = 2$ , and we are left with the same topological phase as before,  $e^{i\theta}$ .

d) The essential identity that we make use of is  $\dot{f}(\mathbf{x}) = \dot{\mathbf{x}} \cdot (\nabla f)(\mathbf{x})$ . We can use this since  $\alpha$  has no explicit dependence on time. Then we have

$$\begin{aligned}
\mathbf{p}_k = \frac{\delta \mathcal{L}}{\delta \dot{\mathbf{q}}_k} &= m\dot{\mathbf{q}}_k - \frac{\theta\hbar}{\pi} \sum_{i < j} \frac{\delta}{\delta \dot{\mathbf{q}}_k} \dot{\alpha}(\mathbf{q}_i - \mathbf{q}_j) \\
&= m\dot{\mathbf{q}}_k - \frac{\theta\hbar}{\pi} \sum_{i < j} \frac{\delta}{\delta \dot{\mathbf{q}}_k} (\dot{\mathbf{q}}_i - \dot{\mathbf{q}}_j) \cdot (\nabla\alpha)(\mathbf{q}_i - \mathbf{q}_j) \\
&= m\dot{\mathbf{q}}_k - \frac{\theta\hbar}{\pi} \sum_{i < j} (\delta_{ik} - \delta_{jk}) (\nabla\alpha)(\mathbf{q}_i - \mathbf{q}_j) \\
&= m\dot{\mathbf{q}}_k - \frac{\theta\hbar}{\pi} \left( \sum_{k < j} (\nabla\alpha)(\mathbf{q}_k - \mathbf{q}_j) - \sum_{j < k} (\nabla\alpha)(\mathbf{q}_j - \mathbf{q}_k) \right) \\
&= m\dot{\mathbf{q}}_k - \frac{\theta\hbar}{\pi} \sum_j (\nabla\alpha)(\mathbf{q}_k - \mathbf{q}_j)
\end{aligned}$$

where in the last line we made use of the fact that  $(\nabla\alpha)(\mathbf{x}) = -(\nabla\alpha)(-\mathbf{x})$ .

To get the second form for the Chern-Simons field, we just take the gradient, plugging in the definition of  $\alpha$ .

$$\alpha = \arctan\left(\frac{y}{x}\right) \Rightarrow (\nabla\alpha) = \frac{1}{x^2 + y^2} \begin{pmatrix} x \\ -y \end{pmatrix}.$$

The second form follows immediately from substitution.

e) Solving the expression for the canonical momentum for  $\dot{\mathbf{q}}_k$ , we obtain

$$\dot{\mathbf{q}}_k = \frac{1}{m} (\mathbf{p}_k + \theta \mathbf{C}(\mathbf{q}_k)).$$

Then the Hamiltonian is given by

$$\mathcal{H} = \sum_k \mathbf{p}_k \dot{\mathbf{q}}_k - \mathcal{L}.$$

Before proceeding with the transformation, it will be helpful to rewrite  $V_{\text{topo}}$  in terms of the Chern-Simons field. We make use of the expressions in part d which translate  $\dot{\alpha}(\mathbf{x})$  into  $\dot{\mathbf{x}} \cdot (\nabla\alpha)(\mathbf{x})$  and the fact that  $\nabla\alpha$  is antisymmetric through the origin. It is also more convenient to use the symmetric form of  $V_{\text{topo}}$ .

$$\begin{aligned}
V_{\text{topo}} &= -\frac{\theta\hbar}{2\pi} \sum_{j,k} (\dot{\mathbf{q}}_j - \dot{\mathbf{q}}_k) \cdot (\nabla\alpha)(\mathbf{q}_j - \mathbf{q}_k) \\
&= -\theta \sum_k \dot{\mathbf{q}}_k \cdot \mathbf{C}(\mathbf{q}_k).
\end{aligned}$$

Now the Hamiltonian becomes

$$\begin{aligned}
\mathcal{H} &= \sum_k \mathbf{p}_k \dot{\mathbf{q}}_k - \frac{m}{2} \dot{\mathbf{q}}_k^2 + \theta \dot{\mathbf{q}}_k \cdot \mathbf{C}(\mathbf{q}_k) + V_0 \\
&= \sum_k \frac{\mathbf{p}_k}{m} (\mathbf{p}_k + \theta \mathbf{C}(\mathbf{q}_k)) - \frac{1}{2m} (\mathbf{p}_k + \theta \mathbf{C}(\mathbf{q}_k))^2 \\
&\quad + \frac{\theta}{m} (\mathbf{p}_k + \theta \mathbf{C}(\mathbf{q}_k)) \cdot \mathbf{C}(\mathbf{q}_k) + V_0.
\end{aligned}$$

Upon simplifying this expression, we arrive at

$$\mathcal{H} = \frac{1}{2m} \sum_k (\mathbf{p}_k + \theta \mathbf{C}(\mathbf{q}_k))^2 + V_0.$$

Now the simple substitution of  $\mathbf{p}_k \rightarrow -i\hbar\nabla_k$  gives us the quantum Hamiltonian

$$\mathcal{H} = \frac{1}{2m} \sum_k (-i\hbar\nabla_k + \theta \mathbf{C}(\mathbf{q}_k))^2 + V_0.$$