Topological Quantum Computing
Solutions 3

July 4, 2006

1. **The Fundamental Group I.**

a) This definition will make sense provided that different choices of class representatives from $[f]$ and $[g]$ all lie in the same unique class $[f \circ g]$. Suppose that $f \sim f'$ and $g \sim g'$. We want to show that $f \circ g \sim f' \circ g'$. Let H_f (H_g) denote a homotopy between f and f' (g and g'). Then let $H_{f\circ g}$ be defined as

$$H_{f\circ g}(t,x) = \begin{cases} H_f(t,2x) & 0 \le x \le 1/2 \\ H_g(t,2x-1) & 1/2 \le x \le 1 \ . \end{cases}$$

Note that we have $H_{f\circ g}(0,x) = f \circ g(x)$ and $H_{f\circ g}(1,x) = f' \circ g'(x)$, so we have constructed a homotopy between $f \circ g$ and $f' \circ g'$. So the class product is independent of the class representative chosen, and this definition makes sense.

b) *Closure:* The product of any two loops is obviously another loop.

Associativity: The products $f \circ (g \circ h)$ and $(f \circ g) \circ h$ describe the same curve in R up to a difference in parameterization, which can obviously be removed by a homotopy. From part a we know that this is sufficient because the product of the classes does not depend of the choice of representative.

Identity: Let $e(x)$ be the constant map, $e(x) = x_0$. The equivalence class $[e]$ is the set of all maps that can be continuously shrunk to the point x_0. For any loop f, $f \circ e$ and $e \circ f$ differ from f only by a change in parameterization which can obviously be removed by a homotopy. Thus, $[f] \circ [e] = [e] \circ [f] = [f]$, so $[e]$ is the identity.

Inverse: Given any loop f, we can define f^{-1} as $f^{-1}(x) = f(1-x)$, so that f^{-1} “untraces” the loop f. We wish to show that $f \circ f^{-1} \sim e$. This is not as obvious since $f \circ f^{-1}$ and e don’t trace out the same curve in R. A homotopy that shows this is given by

$$H(t,x) = \begin{cases} f(2tx) & 0 \le x \le 1/2 \\ f\big(2t(1-x)\big) & 1/2 \le x \le 1 \ . \end{cases}$$

Again, from part a, since we have shown this for some representative, this holds for the class, and the class inverse can be taken to be $[f]^{-1} = [f^{-1}]$.

c) If $f \sim g$ at x_1, then $pfp^{-1} \sim pgp^{-1}$ at x_0. (I’ve dispensed with the notation $\circ$ for product.) Here p^{-1} is the path traversed in the opposite direction: $p^{-1}(x) = p(1-x)$. A required homotopy is given by pHp^{-1} where H is a homotopy between f and g. Thus we have a natural mapping between homotopy classes at x_1 and x_0:

$$p([f]) = [pfp^{-1}].$$

Again, this is independent of the choice of representative from the results of part a.

To show that there is an isomorphism, we need to show that this mapping is injective, surjective, and preserves the group product.

Injective: This is obvious, since for some loop class $[f]$ at x_0, we have that $[p^{-1}fp]$ is the pre-image.

Surjective: We want to show that $p([f]) = p([g]) \Rightarrow [f] = [g]$. From the definition of the map p, $p([f]) = p([g]) \Rightarrow [pfp^{-1}] = [pgp^{-1}]$ from which it follow that $[f] = [p^{-1}pgp^{-1}p] = [g]$.

Homomorphism: We have $(pfp^{-1})(pgp^{-1}) \sim (pfgp^{-1})$, since the $p^{-1}p$ in the middle can be isolated by swapping parentheses at the cost of re-parameterizing, then be removed by a homotopy, and then the resulting curve can be re-parameterized again.

d) We will prove the contrapositive. First assume the path isomorphism is non-unique. Then there exists two paths p and q^{-1} connecting x_0 to x_1 such that $p(\alpha) \neq q^{-1}(\alpha)$ for some class α at x_1. From the definition of the isomorphism mapping, this implies $[pfp^{-1}] \neq [q^{-1}fq]$, where $[f] = \alpha$. We can "push" this equation over to the point x_0 by un-conjugating the path p, which gives $[f] \neq [(qp)^{-1}fqp]$. This is now an equation at x_1. But qp is just a loop at x_1, so this implies $[qp][f] \neq [f][qp]$, which is the criterion for being non-Abelian in the group $\pi_1(R, x_1)$. From part *c*, we conclude that $\pi_1(R)$ is non-Abelian.

Now assume that $\pi_1(R)$ is non-Abelian. From part *c*, we conclude that in particular $\pi_1(R, x_1)$ is non-Abelian. Then there exists two loops at x_1, f and g, with $f \not\sim gfg^{-1}$. Now we construct a new loop at x_1, $\bar{g}$, as follows. Half-way around the loop g, we make a "detour" over the the point x_0 by following a path p connecting $g(1/2)$ to x_0 and then retracing it in reverse, then completing the loop g. More precisely,

$$\bar{g}(x) = \begin{cases} g(2x) & 0 \leq x \leq 1/4 \\ p(4x-1) & 1/4 \leq x \leq 1/2 \\ p^{-1}(4x-2) & 1/2 \leq x \leq 3/4 \\ g(2x-1) & 3/4 \leq x \leq 1\ . \end{cases}$$

Clearly $\bar{g}$ is homotopic to g, so we have $f \not\sim \bar{g}f\bar{g}^{-1}$. We can break $\bar{g}$ into two paths, q and r, where q is the first half of $\bar{g}$ and r is the second half, appropriately re-parameterized, of course, so that $\bar{g} = qr$. Then $f \not\sim qrfr^{-1}q^{-1}$, from which it follows that $q^{-1}([f]) \neq r([f])$, so these two path isomorphisms are not equivalent.

e) Assume that two unbased loops, f and g are in the same conjugacy class of $\pi_1(R, x_0)$. Then there exists a loop h such that $f \sim hgh^{-1}$. With a free homotopy, we are not restricted to remain at the point x_0, so we can simply un-trace the loops h and h^{-1} back into g to show that f and g are freely homotopic.

Now assume that f and g are freely homotopic with free homotopy H. Then $H(t, 0)$ is a path connecting f and g at their respective base points x_f and x_g. Call this path p. Then g is clearly freely homotopic to pgp^{-1}, since we can just untrace p. But note that pgp^{-1} is now a loop at x_f, the base point of f. So we now have that $f \sim pgp^{-1}$, since the total based homotopy is just the free homotopy connecting f and g together with the "detour" of the path p.

This path dependence of the particular isomorphism at two different base points is exactly the path dependence on the flux label that we saw in Preskill's notes. When two different group element fluxons a and b lie in the same conjugacy class, the path taken back to some standard base point can exchange these labels (or more generally change them to any other element in the class). So the elements of a conjugacy class are the indistinguishable particles, and the elements of the class are just multuplets.