7.5 (10 points) Challenge problem (a). Consider the coupled pendulum shown below. Two equal masses $m$ hang from strings of length $l$. The masses are coupled by a spring that has spring constant $k$ and whose unstretched length $b$ is equal to the distance between the strings’ supports.

Throughout this problem you are to use the approximation of small oscillations.

(a) Give the Lagrangian $L$ in terms of the generalized co-ordinates $\theta_1$ and $\theta_2$.

(b) Give the frequencies $\omega_1$ and $\omega_2$ of the normal modes and the (normalized) normal co-ordinates $Q_1$ and $Q_2$, i.e., the co-ordinates such that the the Lagrangian has the form

$$L = \sum_{j=1}^{2} \frac{1}{2} (\dot{Q}_j^2 - \omega_j^2 Q_j^2).$$

Express the relation between the normal co-ordinates and the original co-ordinates as $Q_j = A_{jk} \theta_k$, where $A$ is a matrix. (You should be able to guess the normal co-ordinates and thereby avoid the formal procedure of diagonalizing matrices.)

(c) Let $p_j$ denote the canonical momentum conjugate to $q_j$, and let $P_j$ denote the canonical momentum conjugate to $Q_j$. Find the relation between the two sets of momenta, and express it in terms of the matrix $A$ found in part (b).

(d) Find a generating function $F_2(\theta, P)$ for the canonical transformation from the original co-ordinates and momenta to the normal co-ordinates and momenta.