Lectures 8-24

Hamilton's principle
Hamilton's principle

\[
L = L(\mathbf{q}_1, \ldots, \mathbf{q}_n; \dot{\mathbf{q}}_1, \ldots, \dot{\mathbf{q}}_n; t) = T - V
\]

\[\int_{t_1}^{t_2} \mathrm{d}t \; L = \text{Action} \; I = I[\mathbf{q}_1(t), \ldots, \mathbf{q}_n(t)]
\]

for mechanical systems such that all forces, except conservative forces, are derivable from a potential.

\[\int_{t_1}^{t_2} \mathrm{d}t \; L \rightarrow \text{Note dimensional function.}
\]

Actual motion from \( t_1 \) to \( t_2 \) is such that the action has a stationary value.

\[
0 = \delta I = \delta \int_{t_1}^{t_2} \mathrm{d}t \; L(\mathbf{q}_1, \ldots, \mathbf{q}_n; \dot{\mathbf{q}}_1, \ldots, \dot{\mathbf{q}}_n; t)
\]

\[\text{B.C.: } \mathbf{q}_i(0) = \mathbf{q}_i(t_0) = 0
\]

Physics - what does it mean?

more advanced - variational principle tells us the BCs.

Calculus of variations:

\[
\mathbf{q}_i(t, a) = \mathbf{q}_i(t, 0) + a \gamma_i(t), \quad \text{BC: } \gamma_i(t_0) = 0
\]

\[
\dot{\mathbf{q}}_i(t, a) = \dot{\mathbf{q}}_i(t, 0) + a \dot{\gamma}_i(t)
\]
\[ 0 = \frac{d}{dt} \frac{\delta L}{\delta \dot{q}_i} - \frac{\delta L}{\delta q_i} \]

Integrate by parts:

\[ = \int dt \sum_j \gamma_{ij} \frac{\partial}{\partial \dot{q}_j} \left[ \frac{\partial L}{\partial \dot{q}_j} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) \right] \]

\[ + \sum_j \frac{\partial L}{\partial q_j} \gamma_{ij} \, dq_j \]

\[ = 0 \]

\[ \Rightarrow \quad 0 = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} \quad \text{(Euler-Lagrange equations)} \]

Discuss examples: kinetic energies can be tricky.

Non-holonomic constraints: wheel sliding on wedge.

Holonomic constraints: good idea to write things in terms of.

Generalized coordinates: \( q_i \), \( i = 1, \ldots, N \) \( N - A = n \) independent coordinates.

Constraints:

\[ 0 = f_a (q_{n+1}, \ldots, q_N), \quad a = 1, \ldots, A \]

Extremize:

\[ \int dt \left( L + \sum_a f_a \frac{\partial L}{\partial \dot{q}_a} \right) \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = \sum_a \lambda_a \frac{\partial f_a}{\partial q_i}, \quad i = 1, \ldots, N \]

\[ f_a (q_{n+1}, \ldots, q_N) = 0, \quad a = 1, \ldots, A \]

\[ \Rightarrow \quad \text{Generalized constraint forces:} \]

Adding potentials \( -\lambda_a \) enforces constraints and these are the forces.
Constraints do no work because \( \sum \frac{\delta L}{\delta \dot{q}_i} \dot{q}_i = 0 \)

for any \( \delta q_i \) that lies in the surface \( L = 0 \).

Example: Pendulum

\[
L = \frac{1}{2} m r^2 \dot{\theta}^2 + \frac{1}{2} I \dot{\theta}^2 + m g l \cos \theta
\]

Constraint: \( \dot{r} = 0 = \dot{r} r(\dot{\theta}) = 0 \)

Exteriorize \( \int dt \left( \frac{\delta L}{\delta \dot{r}} \right) \frac{d}{dt} \)

\[
\frac{d}{dt} \left( \frac{\delta L}{\delta \dot{r}} \right) = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right)
\]

\[
\ddot{r} = \frac{m r^2 \dot{\theta}^2 + m g \cos \theta + \lambda}{L}
\]

\[
\frac{d}{dt} \left( \frac{\delta L}{\delta \dot{\theta}} \right) = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right)
\]

\[
\ddot{\theta} = \frac{-m g \sin \theta + \lambda}{m r^2}
\]

\[
L(2 \dot{r} \dot{\theta} + r \ddot{\theta})
\]

Constraint: \( \dot{r} = 0 \)

\[
\frac{\delta L}{\delta \dot{r}} = 2 L \dot{r} \dot{\theta} + r \ddot{\theta} = -m g \sin \theta - mL \dot{\theta}^2
\]

\[
= -T
\]
Conservation laws:

\[ \frac{dL}{dt} = \sum \left( \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i + \frac{\partial L}{\partial q_i} \right) + \frac{\partial L}{\partial t} \]

\[ \frac{d}{dt} \left( \sum \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) + \frac{\partial L}{\partial q_i} = \frac{d}{dt} \left( \sum \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) \]

\[ \frac{d}{dt} \left( L - \sum \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) = \frac{\partial L}{\partial t} \]

If no explicit time dependence,

\[ \frac{dh}{dt} = 0, \quad h = \sum \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L = (\text{Jacobi's (integrated)}) \]

If \( L = T - V \) and \( T = \sum \frac{1}{2} m_\alpha \dot{q}_\alpha \dot{q}_\alpha \), \( \frac{\partial L}{\partial \dot{q}_i} = \sum m_\alpha \ddot{q}_\alpha \)

\[ \sum \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i = \sum m_\alpha \ddot{q}_\alpha \dot{q}_i \]

\[ \text{so} \quad h = T + V \]

\( \text{Generalized} \)

\( \text{Covariant momentum} \quad p_i = \frac{\partial L}{\partial \dot{q}_i} \)

\( \text{Conjugate} \)

\( \text{Lagrange equations:} \quad \ddot{q}_i = \frac{\partial L}{\partial \dot{q}_i} \)

\( \text{If} \quad L \text{ is independent of } \dot{q}_i \text{ (cyclic),} \quad p_i = 0 \)

Relation to Symmetries
Assume \( T = \sum_{c} \frac{1}{2} m_c \dot{\mathbf{r}}_c \), \( \mathbf{V} = \mathbf{V}(q_1, \ldots, q_n) \).

1. **Translations:** Suppose \( \dot{q}_i \) corresponds to translating entire system (CM) along \( \dot{\mathbf{r}}_i \).

\[
\mathbf{P}_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = \sum_c \mathbf{r}_c \times \mathbf{\dot{r}}_c = \sum_c m_c \mathbf{\dot{r}}_c \cdot \mathbf{n} = \mathbf{P} \cdot \mathbf{n}
\]

\[
\dot{\mathbf{r}}_c = \frac{\partial \mathbf{r}_c}{\partial \dot{q}_i} \cdot \dot{q}_i.
\]

\[
\mathbf{F}_i = \sum_c \mathbf{r}_c \times (m_c \mathbf{\dot{r}}_c) \cdot \mathbf{n} = \mathbf{F} \cdot \mathbf{n}
\]

\( T \) cannot depend on CM position.

2. **Rotations:** Suppose \( \dot{q}_i \) corresponds to rotating entire system about \( \mathbf{n} \).

\[
\mathbf{P}_i = \frac{\partial L}{\partial \dot{\mathbf{r}}_i} = \frac{\partial T}{\partial \dot{\mathbf{r}}_i} = \sum_c \mathbf{r}_c \times \mathbf{\dot{r}}_c = \mathbf{n} \times \sum_c m_c \mathbf{\dot{r}}_c \times (m_c \mathbf{\dot{r}}_c) \cdot \mathbf{n} \cdot \mathbf{L}
\]

\[
\dot{\mathbf{r}}_c = \frac{\partial \mathbf{r}_c}{\partial \dot{q}_i} \cdot \dot{q}_i.
\]

\[
\dot{\mathbf{r}}_c = \frac{\partial \mathbf{r}_c}{\partial \dot{q}_i} \cdot \dot{q}_i.
\]

\[
\mathbf{F}_i = \sum_c \mathbf{r}_c \times \mathbf{F}_c = n \cdot \mathbf{N}
\]

\( T \) cannot depend on Orientation.
Motion of a charged particle

Efficient vector calculus:

Right-handed orthonormal Cartesian basis:

\[ \vec{e}_1 = \vec{e}_x, \quad \vec{e}_2 = \vec{e}_y, \quad \vec{e}_3 = \vec{e}_z \]

\[ \vec{e}_j \cdot \vec{e}_k = \delta_{jk} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases} \]

\[ \vec{e}_j \times \vec{e}_k = \sum_{l} \epsilon_{jkl} \vec{e}_l = \vec{e}_l \epsilon_{jkl} \]

Summation convention.

Totally antisymmetric symbol — consequences of antisymmetry.

Cyclic property.

Vector: \[ \vec{A} = A_j \vec{e}_j \]

Vector algebra:

\[ \vec{A} \cdot \vec{B} = A_j B_k \vec{e}_j \cdot \vec{e}_k = A_j B_j \]

\[ \vec{A} \times \vec{B} = A_j B_k \vec{e}_j \times \vec{e}_k = A_k \epsilon_{ijk} \vec{B}_i \]

\[ \vec{A} \cdot (\vec{B} \times \vec{C}) = \epsilon_{jkl} A_j B_k C_l \]

\[ \vec{A} \times (\vec{B} \times \vec{C}) = A_j B_k C_l \vec{e}_j \times (\vec{e}_k \times \vec{e}_l) \]

\[ \epsilon_{ijk} \epsilon_{klm} = \delta_{jm} \delta_{lk} - \delta_{jl} \delta_{km} \]

\[ \epsilon_{ijk} \epsilon_{jlm} = \vec{e}_n (A_j B_m C_k - A_j B_k C_m) \]

\[ = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \]
Vector calculus: \[ \nabla = \vec{e}_x \frac{\partial}{\partial x} \]

\[ \nabla f = \vec{e}_x \frac{\partial f}{\partial x} \]

\[ \nabla \cdot \vec{A} = \vec{e}_x \frac{\partial}{\partial x} \vec{e}_k \hat{A}_k \cdot \vec{A}_{j;k} \]

\[ \nabla \times \vec{A} = \vec{e}_x \frac{\partial}{\partial x} \vec{e}_k \vec{A}_k = \vec{e}_x \epsilon_{jkl} \vec{A}_{j;k} = \vec{e}_x \epsilon_{jkl} \vec{A}_{j;k} \]

\[ \vec{B} = \nabla \times \vec{A} \]

\[ \vec{E} = - \nabla \phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \]

\[ E_j = \phi_{,j} - \frac{1}{c} A_{,jt} \]

\[ B_j = \epsilon_{jkl} A_{,kt} \]

\[ \nabla \times \vec{B} = \vec{e}_x \epsilon_{jkl} \hat{V}_{j} \vec{B}_{k} \]

\[ = \vec{e}_x \epsilon_{jkl} \epsilon_{jmn} \hat{V}_j A_{nm} \]

\[ = \vec{e}_x \epsilon_{jkl} \hat{V}_j \vec{A}_{nm} \]

\[ = \vec{e}_x \hat{V}_j (\vec{A}_{jn} - \vec{A}_{jn}) \]

\[ m \ddot{x}_j = - q \dot{\phi}_{,j} - \frac{q}{c} A_{,jt} + \frac{q}{c} \hat{V}_j (\vec{A}_{jn} - \vec{A}_{jn}) \]

Lagrangian: \[ L = \frac{1}{2} m \dot{x}_j \dot{x}_j - q \dot{\phi} + \frac{q}{c} \vec{\nabla} \cdot \vec{A} \]

\[ = \frac{1}{2} m \dot{x}_j \dot{x}_j - q \phi (\vec{v}_j \dot{t}) + \frac{q}{c} \vec{\nabla} \cdot \vec{A} (\vec{v}_j \dot{t}) \]
\[ \frac{\partial L}{\partial \ddot{x}_j} = m \dddot{x}_j + \frac{g}{c} A_j (\dddot{x}_j) \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_j} \right) = m \dddot{x}_j + \frac{g}{c} A_{jkt} + \frac{g}{c} A_{jk} \ddot{x}_k \]

\[ = \frac{\partial L}{\partial \dot{x}_j} \]

\[ = -g \ddot{\phi}_j + \frac{g}{c} \dddot{x}_k A_{kj} \]

\[ m \dddot{x}_j = -g \ddot{\phi}_j - \frac{g}{c} A_{jst} + \frac{g}{c} \dddot{x}_k (A_{k,j} - A_{j,k}) \]