Consider a quantum system with Hamiltonian $H$. The stationary states are the eigenstates of $H$: $H|E\rangle = E|E\rangle$. The state vector $|\psi(t)\rangle$ can be written in terms of an arbitrary basis, $\{|j\rangle\}$, as

$$|\psi(t)\rangle = \sum_j |j\rangle \langle j|\psi(t)\rangle = \sum_j \eta_j(t)|j\rangle,$$

where the “original coordinates” are the amplitudes $\eta_j(t) = \langle j|\psi(t)\rangle$. The state vector can also be written in terms of the energy eigenbasis as

$$|\psi(t)\rangle = \sum_E |E\rangle \langle E|\psi(t)\rangle = \sum_E \zeta_E(0)e^{-iEt/h}|E\rangle,$$

where the amplitudes

$$\zeta_E(t) = \langle E|\psi(t)\rangle = \langle E|e^{-iHt/h}|\psi(0)\rangle = e^{-iEt/h}\langle E|\psi(0)\rangle = e^{-iEt/h}\zeta_E(0)$$

are the normal coordinates because they oscillate sinusoidally in time.

The original coordinates are related to the normal coordinates by

$$\eta_j(t) = \langle j|\psi(t)\rangle = \sum_E \langle j|E\rangle \langle E|\psi(t)\rangle = \sum_E a_{JE}\zeta_E(0)e^{-iEt/h}.$$

The energy eigenstates are the normal modes; they are related to the basis $|j\rangle$ by

$$|E\rangle = \sum_j |j\rangle \langle j|E\rangle = a_{JE}|j\rangle.$$

The matrix elements $a_{JE}$ give the “shape” of normal mode $E$ in the original basis.

The only difference with what we are doing in linear mechanical systems is that the variables are real instead of complex and thus obey second-order, linear differential equations in time, instead of the first-order, linear Schrödinger equation. Once one is past that difference, the whole thing is identical. If we were to master the bra-ket notation for the real vector spaces in mechanics, the question of which order to write the indices in and whether to use a transformation matrix or its transpose would disappear as questions, since the bra-ket notation takes care of all that without any need for thinking.