2.3 Qubit rotations. An arbitrary unitary operator in a two-dimensional vector space can be written in the form

\[ U = \exp(i\delta - in \cdot \sigma \theta / 2) = e^{i\delta} e^{-in \cdot \sigma \theta / 2}. \]

The phase \( e^{i\delta} \) produces a global phase change, so we can discard it and write the general unitary operator as

\[ U_R = e^{-in \cdot \sigma \theta / 2}. \]

(a) What is the eigendecomposition of \( U_R \)?

(b) Show that

\[ U_R = 1 \cos(\theta / 2) - in \cdot \sigma \sin(\theta / 2). \]

(c) Show that

\[ U_R^\dagger \sigma U_R = n(n \cdot \sigma) - n \times (n \times \sigma) \cos \theta + n \times \sigma \sin \theta \]
\[ = \sigma \cos \theta + n(n \cdot \sigma)(1 - \cos \theta) + n \times \sigma \sin \theta \equiv R_n(\theta)\sigma. \]

Here \( R_n(\theta) \) is the 3-dimensional orthogonal matrix that describes a rotation by angle \( \theta \) about axis \( n \).

(d) Use the result of part (c) to show that \( U_R \) rotates any state \( |m\rangle \), i.e., that

\[ U_R|m\rangle = e^{i\phi(R,m)}|Rm\rangle, \]

where \( \phi(R,m) \) is a phase.

(e) Show that the unitary operator \( n \cdot \sigma \) produces a 180° rotation about \( n \).

(f) The Hadamard transform,

\[ H \equiv ie^{-in \cdot \sigma \pi / 2}, \]

where \( n = (e_x + e_z)/\sqrt{2} \), rotates by 180° about the axis midway between the \( x \) and \( z \) axes. Show that the Hadamard transform has the matrix representation

\[ H \leftrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \]