7.1 Cloning and the isotropic POVM. The isotropic POVM for a $D$-dimensional Hilbert space has a POVM element, $dE_{|\phi\rangle} = \alpha |\phi\rangle \langle \phi | d\Pi_{|\phi\rangle}$, for every ray $|\phi\rangle$ in projective Hilbert space, where $d\Pi_{|\phi\rangle}$ is the unitarily invariant measure on projective Hilbert space and $\alpha$ is a positive constant, thus implying that all outcomes are equally weighted. The POVM satisfies a completeness relation

$$I = \int dE_{|\phi\rangle} = \alpha \int d\Pi_{|\phi\rangle} |\phi\rangle \langle \phi | .$$

If the system is in state $\rho$, the probability to get outcome $|\phi\rangle$ in a measurement of the isotropic POVM is

$$dp(|\phi\rangle | \rho) = \text{tr}(\rho dE_{|\phi\rangle}) = \alpha d\Pi_{|\phi\rangle} \langle \phi | \rho | \phi \rangle .$$

To do this problem, we need to be able to do at least one integral over the invariant measure. The unitarily invariant line element on projective Hilbert space, called the Fubini-Studi metric, measures lengths in terms of Hilbert-space angle: the distance $d\gamma$ between a normalized vector $|\phi\rangle$ and a nearby normalized vector $|\phi'\rangle = |\phi\rangle + |d\phi\rangle$ is given by $\cos d\gamma = |\langle \phi | \phi' \rangle| = |1 + \langle \phi | d\phi \rangle|$. Hilbert-space angle is not changed by global phase changes, confirming that we are dealing with rays in projective Hilbert space. The Fubini-Studi line element is given by

$$d\gamma^2 = \sin^2 d\gamma = 1 - \cos^2 d\gamma = -2\text{Re}(\langle \phi | d\phi \rangle) - |\langle \phi | d\phi \rangle|^2 .$$

Normalization of $|\phi'\rangle$ requires that

$$0 = \langle \phi' | \phi' \rangle - 1 = 2\text{Re}(\langle \phi | d\phi \rangle) + \langle d\phi | d\phi \rangle ,$$

which gives

$$d\gamma^2 = \langle d\phi | d\phi \rangle - |\langle \phi | d\phi \rangle|^2 = \langle d\phi | d\phi \rangle - (\text{Im}(\langle \phi | d\phi \rangle))^2 = \langle d\phi_\perp | d\phi_\perp \rangle ,$$

where $|d\phi_\perp\rangle = |d\phi\rangle - |\phi\rangle \langle \phi | d\phi \rangle$ is the projection of the small displacement $|d\phi\rangle$ orthogonal to $|\phi\rangle$.

The normalized vectors in a $D$-dimensional Hilbert space make up the sphere of unit radius, $S_{2D-1}$, in $2D$ real dimensions. The contribution $\langle d\phi | d\phi \rangle$ to the Fubini-Studi line element is the standard metric on this unit sphere. The quantity $\langle \phi | d\phi \rangle$ is the component of the small displacement along $|\phi\rangle$: the real part describes changes in normalization, and the imaginary part describes changes in phase. The real part disappears from the line element because of the normalization constraint; the square of the imaginary part is subtracted away to remove the contribution of phase changes, because a global phase change does not change the Hilbert-space angle between two vectors.
Given a particular normalized vector $|\psi\rangle$, any other normalized vector can be written as

$$|\phi\rangle = e^{i\delta}(\cos \theta |\psi\rangle + \sin \theta |\chi\rangle) ,$$

where $\delta$ is a global phase, $\theta$ is a “polar angle” in the range $0 \leq \theta \leq \pi/2$, and $|\chi\rangle$ is a normalized vector orthogonal to $|\psi\rangle$. We can get rid of the global phase freedom by choosing $\delta = 0$, thus working with rays in projective Hilbert-space. A small change in $|\phi\rangle$ takes the form

$$|d\phi\rangle = d\theta(-\sin \theta |\psi\rangle + \cos \theta |\chi\rangle) + \sin \theta |d\chi\rangle ,$$

which gives

$$\langle \phi |d\phi\rangle = \sin^2 \theta \langle \chi |d\chi\rangle ,$$

$$\langle d\phi |d\phi\rangle = d\theta^2 + \sin^2 \theta \langle d\chi |d\chi\rangle .$$

The resulting line element is

$$d\gamma^2 = d\theta^2 + \sin^2 \theta \left(\langle d\chi |d\chi\rangle - \sin^2 \theta |\langle d\chi |d\chi\rangle|^2\right)$$

$$= d\theta^2 + \sin^2 \theta \left(\langle d\chi_{\perp} |d\chi_{\perp}\rangle + \cos^2 \theta |\langle \chi |d\chi\rangle|^2\right) ,$$

where $d\chi_{\perp} = |d\chi\rangle - |\chi\rangle \langle \chi |d\chi\rangle$ is the projection of $|d\chi\rangle$ orthogonal to $|\chi\rangle$.

The line element $\langle d\chi |d\chi\rangle - \sin^2 \theta |\langle \chi |d\chi\rangle|^2 = \langle d\chi_{\perp} |d\chi_{\perp}\rangle + \cos^2 \theta |\langle \chi |d\chi\rangle|^2$ is the standard metric on the unit sphere, $S_{2D-3}$, in $2D - 2$ real dimensions, except that along one real dimension, corresponding to phase changes of $|\chi\rangle$, lengths are scaled by a factor $\cos \theta$. The $\sin^2 \theta$ in the line element means that lengths along all $2D - 3$ real dimensions of $S_{2D-3}$ are scaled by a factor of $\sin \theta$. Thus the integration measure that goes with the Fubini-Studi metric is

$$d\Gamma|\phi\rangle = \sin^{2D-3} \theta \cos \theta \ d\theta \ dS_{2D-3} ,$$

where $dS_{2D-3}$ is the standard measure on the unit sphere $S_{2D-3}$. This form of the integration measure is useful for doing integrals over functions of $|\langle \phi |\psi\rangle| = \cos \theta$.

In doing this problem, you should never have to calculate an explicit form for the area of a sphere. Instead, the area $S_{2D-3}$ of $S_{2D-3}$ can be left as a normalization constant, whose value cancels out of the ultimate answer to part (c).

(a) Using the completeness relation, determine the value of the positive constant $\alpha$.

(b) Find the value of the integral

$$\int d\Gamma|\phi\rangle |\langle \phi |\psi\rangle|^2 .$$

(c) One strategy for approximate cloning of an arbitrary state $|\psi\rangle$ is to measure the isotropic POVM and then make copies of the result $|\phi\rangle$. Find the average squared fidelity of the copies.