1.3. Probability simplex: \( \mathbf{1} = \sum_{j=1}^{D} P_j \) \( \rightarrow \) hyperplane in \( \mathbb{R}^D \)

\[ P_j \geq 0, \ j = 1, \ldots, D \ \rightarrow \ \text{all positive} \]

\( \mathbb{R}^D \)-convex.

(a) \( D = 2 \)

\( \text{Simplex} \)

\[ P_1 = 0 \]
\[ P_1 + P_2 + P_3 = 1 \]

D = 3

\( \text{Simplex (equilateral triangle)} \)

D = 4

\( \text{Tetrahedron} \)

\[ P_1 = 1 \]
\[ P_1 + P_2 = 1 \]
\[ P_2 + P_3 = 0 \]
\[ P_2^* P_3 = 0 \]

Regular polyhedra with edges of "length" \( \sqrt{2} \). The boundaries are probability simplices for all numbers of alternatives between 2 and \( D-1 \), ranging from the edges (2 alternatives) to the faces (\( D-1 \) alternatives).

Vertices: one unit probability, rest zero
Edges: two probabilities zero, simplex for other two
Faces: one probability zero, simplex for other three
(b) \( G_j = P_j + \delta P_j \)

\[
1 = \sum_{j} G_j = \sum_{j} P_j + \sum_{j} \delta P_j \Rightarrow \sum_{j} \delta P_j = 0
\]

Relative information:

\[
H(\bar{g}||\bar{p}) = \sum_{j} g_j \log \frac{g_j}{P_j} = -H(\bar{p}) - \sum_{j} g_j \log P_j
\]

Expand:

\[
H(\bar{g}||\bar{p}) = \sum_{j} (P_j + \delta P_j) \log (1 + \frac{\delta P_j}{P_j})
\]

\[
= \frac{\ln (1 + \frac{\delta P_j}{P_j})}{1 + \frac{\delta P_j}{P_j}}
\]

\[
= \frac{1}{1 + \frac{\delta P_j}{P_j}} \left( \frac{\delta P_j}{P_j} - \frac{1}{2} \left( \frac{\delta P_j}{P_j} \right)^2 + \ldots \right)
\]

\[
= \frac{1}{\ln 2} \sum_{j} \delta P_j - \frac{1}{2} \frac{\delta P_j^2}{P_j} + \frac{\delta P_j^2}{P_j} + \ldots
\]

\[
H(\bar{g}||\bar{p}) = \frac{1}{\ln 2} \sum_{j} \frac{\delta P_j^2}{P_j} = \frac{2}{\ln 2} \sum_{j} \frac{\delta P_j^2}{4P_j}
\]

(c) \( y_j = \sqrt{\frac{P_j}{\delta P_j}} \)

Probability simplex:

\[
\sum_{j} P_j = \sum_{j} y_j^2 = 1 \leftarrow \text{sphere of unit radius}
\]

\( P_j > 0 \rightarrow y_j > 0 \leftarrow \text{respects to portion of sphere in all positive } \mathbb{R}^2 \text{ - out}
\]

The metric on the sphere is the standard round metric induced by this flat metric on Euclidean space.