Solution 3.2.

We want to show that \( \rho \) is a pure state if and only if

\[
\rho = \rho^\dagger, \quad \text{tr}(\rho^2) = 1, \quad \text{tr}(\rho^3) = 1.
\]

A pure state clearly satisfies these properties, so all we need to show is sufficiency.

Since \( \rho \) is Hermitian, it has an eigendecomposition, \( \rho = \sum_j \lambda_j |e_j\rangle\langle e_j| \), where the eigenvalues \( \lambda_j \) are real (we cannot assume they are nonnegative). The condition,

\[
1 = \text{tr}(\rho^2) = \sum_j \lambda_j^2,
\]

shows that \( 0 \leq \lambda_j^2 \leq 1 \) or, equivalently, that \( 0 \leq |\lambda_j| \leq 1 \). We can use this result and the trace conditions to write

\[
1 = \text{tr}(\rho^3) = \sum_j \lambda_j^3 \leq \sum_j |\lambda_j| |\lambda_j^2| \leq \sum_j \lambda_j^2 = 1.
\]

The bookending inequalities imply that the inequalities must be saturated. The first inequality is saturated if and only if all the eigenvalues are nonnegative, giving us \( 0 \leq \lambda_j \leq 1 \), and the second if and only if there is one eigenvalue equal to 1, with all the others equal to 0. Thus \( \rho \) is a pure state.