Problem 1: Landé Projection Theorem (10 Points)

The Landé Projection Theorem (LPT) is a special case of the Wigner-Eckart Theorem for the case that the initial and final state are the same angular momentum. It states:

\[ \langle \alpha'; j' m'| \hat{V} | \alpha; j m \rangle = \frac{j' m'}{j(j+1)} \langle \alpha'; j \hat{J} \cdot \hat{V} | \alpha; j \rangle \langle j m' | \hat{J} | j m \rangle, \]

where \( \hat{V} \) is a vector operator w.r.t. \( \hat{J} \).

(a) Give a geometric interpretation of this in terms of a vector picture.

(b) To prove this theorem, take the following steps (do not give verbatim, Sakurai’s derivation):

(i) Show that

\[ \langle \alpha'; j', m', \hat{J} \cdot \hat{V} | \alpha; j, m \rangle = \langle \alpha'; j \hat{J} \cdot \hat{V} | \alpha; j \rangle \langle j m' | \hat{J} | j m \rangle, \]

independent of \( m \).

(ii) Use this to show,

\[ \langle \alpha; j \hat{J} \cdot \hat{V} | \alpha; j \rangle = j(j+1) \]

independent of \( \alpha \).

(iii) Show that

\[ \langle j m' | q j m \rangle = \langle j m' | \hat{J} q j m \rangle / \sqrt{j(j+1)} \]

(iv) Put it all together to prove the LPT.

(c) As an application of the LPT, consider the Zeeman interaction for hydrogen.

\[ \hat{H}_{\text{int}} = -\hat{\mu} \cdot \mathbf{B}, \]

where the magnetic dipole operators is \( \hat{\mu} = -\mu_b (g_l \hat{L} + g_s \hat{S}) \), with \( g_l = 1, g_s = 2 \).

If the Zeeman effect is small compared to the fine-structure, but ignoring hyperfine structure, use the LPT to show that in a state \( nL_j \), the magnetic moment has the form,

\[ \hat{\mu} = -g_j \mu_B \hat{J}, \]

where \( g_j = \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)} \) is known as the Landé g-factor.

Hint: Use \( \mathbf{J} \cdot \mathbf{L} = L^2 + \frac{1}{2}(J^2 - L^2 - S^2) \) and \( \mathbf{J} \cdot \mathbf{S} = S^2 + \frac{1}{2}(J^2 - L^2 - S^2) \).

(d) Estimate the magnetic field strength at which the Zeeman interaction becomes on the order of the fine structure splitting between the \( 2p_{3/2} \) and \( 2p_{3/2} \) state in hydrogen.
Problem 2: Natural lifetimes of Hydrogen (10 points)

Though in the absence of any perturbation, an atom in the excited state will stay there forever (it is a stationary state), in reality it will “spontaneously decay” to the ground state. Fundamentally this occurs because the atom is always perturbed by ‘vacuum fluctuations’ in the electro-magnetic field. We will find later in the semester, the spontaneous emission rate on a dipole allowed transition from initial excited state $|\psi_e\rangle$ to all allowed grounds states $|\psi_g\rangle$ is,

$$\Gamma = \frac{4}{3h} k^3 \sum_g \left| \langle \psi_g | \hat{d} | \psi_e \rangle \right|^2,$$

where $k = \omega_{eg} / c$ is the emitted photon’s wave number.

Consider now hydrogen including fine-structure. For a given sublevel, the spontaneous emission rate is

$$\Gamma_{(nLMJ) \rightarrow (n'L'J')} = \frac{4}{3h} k^3 \sum_{M_J} \left| \langle n'L'J'M'_J | \hat{d} | nLJM_J \rangle \right|^2.$$

(a) Show that the spontaneous emission rate is independent of the initial $M_J$. Explain this result physically.

(b) Calculate the lifetime ($\tau = 1/\Gamma$) of the $2p_{1/2}$ state in seconds.
**Problem 3: Light-shift for multilevel atoms (10 points)**

Consider a general monochromatic electric field $\mathbf{E}(\mathbf{x},t) = \text{Re}(\mathbf{E}(\mathbf{x})e^{-i\omega t})$, driving an atom near resonance on the transition $|g;J_g\rangle \rightarrow |e;J_e\rangle$, where the ground and excited manifolds are each described by some total angular momentum $J$ with degeneracy $2J+1$. The generalization of the AC-Stark shift is the light-shift operator acting on the $2J_g+1$ dimensional ground manifold:

$$\hat{V}_{LS}(\mathbf{x}) = -\frac{1}{4} \mathbf{E}^*(\mathbf{x}) \cdot \hat{\alpha} \cdot \mathbf{E}(\mathbf{x}).$$

Here $\hat{\alpha} = -\frac{\hat{d}_{ge} \hat{d}_{eg}}{\hbar \Delta}$ is the atomic polarizability tensor operator, where $\hat{d}_{ge} \equiv \hat{P}_e \hat{d} \hat{P}_g$ is the dipole operator, projected between the ground and excited manifolds; the projector onto the excited manifold is, $\hat{P}_e = \sum_{M_e=J_e}^J \sum_{M_g=-J_g}^{J_g} |e;J_e,M_e\rangle \langle e;J_e,M_e|$, and similarly for the ground.

(a) By expanding the dipole operator in the spherical basis, **show** that the polarizability operator can be written,

$$\hat{\alpha} = \hat{\alpha} \left( \sum_{q,M_q} C_{M_q}^{M_e+q} |g;J_g,M_g\rangle \langle g;J_g,M_g| \hat{e}_q^* + \sum_{q,M_q} C_{M_q}^{M_e+q} \hat{e}_q \langle g;J_g,M_g + q - q_0 \rangle \langle g;J_g,M_g | \hat{q} \right)$$

where $\hat{\alpha} \equiv -\frac{\langle e;J_e|\hat{d}|g;J_g\rangle^2}{\hbar \Delta}$ and $C_{M_q}^{M_e+q} \equiv \langle J_e,M_e|\hat{q} J_g,M_g \rangle$.

(b) Consider a polarized plane wave, with complex amplitude of the form, $\mathbf{E}(\mathbf{x}) = E_i \hat{e}_x e^{i \omega x}$ where $E_i$ is the amplitude and $\hat{e}_x$ the polarization (possibly complex). For an atom driven on the transition $|g;J_g=1\rangle \rightarrow |e;J_e=2\rangle$ and the cases (i) linear polarization along $z$, (ii) positive helicity polarization, (iii) linear polarization along $x$, **find** the eigenvalues and eigenvectors of the light-shift operator. Express the eigenvalues in units of $V_i = -\frac{1}{4} \alpha |E_i|^2$. Please **comment** on what you find for cases (i) and (iii).

**Repeat** for $|g;J_g=1/2\rangle \rightarrow |e;J_e=3/2\rangle$ and **comment**.

(c) A deeper insight into the light-shift potential can be seen by expressing the polarizability operator in terms of irreducible tensors. **Show** that we get the sum of scalar, vector, and rank-2 irreducible tensor interaction,

$$\hat{V}_{LS} = -\frac{1}{4} \left( |\mathbf{E}(\mathbf{x})|^2 \hat{\alpha}^{(0)} + (\mathbf{E}^*(\mathbf{x}) \times \mathbf{E}(\mathbf{x})) \cdot \hat{\alpha}^{(1)} + \mathbf{E}^*(\mathbf{x}) \cdot \hat{\alpha}^{(2)} \cdot \mathbf{E}(\mathbf{x}) \right)$$

where $\hat{\alpha}^{(0)} = \frac{\hat{d}_{ge} \hat{d}_{eg}}{-3\hbar \Delta}$, $\hat{\alpha}^{(1)} = \frac{\hat{d}_{ge} \times \hat{d}_{eg}}{-2\hbar \Delta}$, $\hat{\alpha}^{(2)} = \frac{1}{-\hbar \Delta} \left( \frac{\hat{d}_{ge} \hat{d}_{ge} \hat{d}_{ge} + \hat{d}_{ge} \hat{d}_{ge} \hat{d}_{ge} - \hat{d}_{ge} \hat{d}_{ge} \delta^{(2)}}{2} \right)$. 