
5. A system whose state space is \( \mathcal{E} \), has for its wave function:

\[
\psi(x, y, z) = N(x + y + z) e^{-r^2/a^2}
\]

where \( a \), which is real, is given and \( N \) is a normalization constant.

a. The observables \( L_x \) and \( L^2 \) are measured; what are the probabilities of finding 0 and \( 2\hbar^2 \)? Recall that:

\[
Y_l^0(\theta, \varphi) = \frac{\sqrt{3}}{\sqrt{4\pi}} \cos \theta
\]

b. If one also uses the fact that:

\[
Y_l^\pm(\theta, \varphi) = \mp \frac{\sqrt{3}}{\sqrt{8\pi}} \sin \theta e^\pm i\varphi
\]

is it possible to predict directly the probabilities of all possible results of measurements of \( L^2 \) and \( L_z \) in the system of wave function \( \psi(x, y, z) \)?


6. Consider a system of angular momentum \( l = 1 \). A basis of its state space is formed by the three eigenvectors of \( L_z \): \( |+1\rangle \), \(|0\rangle\), \(|-1\rangle\), whose eigenvalues are, respectively, +\( \hbar \), 0, and −\( \hbar \), and which satisfy:

\[
L_\pm |m\rangle = \hbar \sqrt{2} |m \pm 1\rangle \\
L_+ |1\rangle = L_- | -1\rangle = 0
\]

This system, which possesses an electric quadrupole moment, is placed in an electric field gradient, so that its Hamiltonian can be written:

\[
-\hat{H} = \frac{\omega_0}{\hbar} (L_+^2 - L_-^2)
\]

where \( L_+ \) and \( L_- \) are the components of \( L \) along the two directions \( Ox \) and \( Oz \) of the \( xOz \) plane which form angles of 45° with \( Ox \) and \( Oz \); \( \omega_0 \) is a real constant.
Problem 3: The rigid rotator (15 points)
Consider a dumbbell model of a diatomic molecule, with two masses attached rigidly to a massless rod of length \( d \).

(a) Assuming \( d \) cannot change, and its center of mass does not change, show that the Hamiltonian is

\[ \hat{H} = \frac{\hat{L}^2}{2l^2} \]

where \( \hat{L}^2 \) is the squared angular momentum and \( l \) is the moment of inertia of the masses for rotation perpendicular to the dumbbell.

(b) What are the energy levels of the system? What is their degeneracy? What are the energy eigenfunctions? Sketch a level diagram.

(c) Diatomic nitrogen, with \( d=100 \text{ pm} \). Suppose a quantum jump occurs from level denoted by quantum number \( l+1 \) to \( l \). What is the wavelength of the emitted?

(d) Suppose you measured the spectrum of emitted light between different transitions. Explain how you would you it to measure the moment of inertia of the molecule.