Physics 522, Spring 2016
Problem Set #5
Due: Thursday Feb. 25, 2016 @ 5PM

Problem 3: The Finite Spherical Well (20 points)

Consider a spherically symmetric potential, \( V(r) = \begin{cases} -V_0 & 0 < r < a \\ 0 & r > a \end{cases} \). Along the radial coordinate, due to the boundary condition at \( r=0 \), this is just the half-finite well we studied in Problem Sets 5 and 6.

(a) For \( E<0 \), the solutions to the T.I.S.E. are bound states. Let \( E = -E_b \). Making the ansatz for the stationary state wave functions \( \psi_{E,J,m}(r,\theta,\phi) = R_{E,J}(r)Y_{l}^{m}(\theta,\phi) \), show that the radial function must have the form,

\[
R_{E,J}(r) = \begin{cases} 
A j_l(k_1r) & 0 < r < a \\
A \frac{j_l(k_1r) - h_l^{(1)}(ik\alpha)}{h_l^{(1)}(\alpha)} & r > a
\end{cases}, \quad \text{where} \quad k_1 = \sqrt{\frac{2m}{\hbar^2} (V_0 - E_b)}, \quad \kappa = \sqrt{\frac{2m}{\hbar^2} E_b}.
\]

How would you determine \( A \)?

(b) Show that the binding energies are determined by the transcendental equation

\[
\begin{bmatrix}
\frac{d}{dr}(rj_l(k_1r)) \\
\frac{j_l(k_1r)}{rj_l(k_1r)}
\end{bmatrix}_{r=a} = \begin{bmatrix}
\frac{d}{dr}\left( r h_l^{(1)}(ik\kappa) \right) \\
\frac{h_l^{(1)}(ik\kappa)}{r h_l^{(1)}(ik\kappa)}
\end{bmatrix}_{r=a}.
\]

Does this reduce to the expected solution of s-states (i.e. \( l = 0 \)).

(c) Now consider the unbound states. We seek the scattering phase shift for the asymptotic incoming and outgoing partial waves, as discussed in Lecture.
Show that the phase satisfies the equation

\[
\left( \frac{r_j(qr)}{d/dr[r_j(qr)]} \right)_{r=a} = \left( \frac{r(\cos(\delta_l/2)j_{l}(kr) - \sin(\delta_l/2)n_{l}(kr))}{d/dr[r(\cos(\delta_l/2)j_{l}(ka) - \sin(\delta_l/2)n_{l}(kr))] \right)_{r=a},
\]

where \( k = \sqrt{\frac{2m}{\hbar^2}} E \) and \( q = \sqrt{\frac{2m}{\hbar^2}} (E + V_0) \).

Check that this limits to the expected result for s-wave (\( l=0 \)).

**Problem 2: The 3D Isotropic Simple Harmonic Oscillator.** (20 points)

Consider a particle of mass \( m \) moving in a three dimensional isotropic SHO, with frequency \( \omega \).

(a) Since the problem is separable in Cartesian coordinates, show that the energy eigenvalues are

\[
E_n = \hbar \omega (n + 3/2), \quad \text{where } n = 0,1,2,... \quad \text{Show that the degeneracy is } g_n = \frac{(n+1)(n+2)}{2}.
\]

(b) The degeneracy is of course stemming from the rotational symmetry of the problem. Let us now seek simultaneous eigenfunctions of \( \{ \hat{H}, \hat{L}_r^2, \hat{L}_z \} \) and separate in spherical coordinates, so that the wave function is \( \psi_{n,l,m}(r,\theta,\phi) = \frac{u_{n,m}(r)}{r} Y_{l,m}(\theta,\phi) \). Defining the usual dimensionless variables \( \xi \equiv r / r_c, \quad \varepsilon \equiv E / \hbar \omega \), (where \( r_c = \sqrt{\hbar / m \omega} \)), write the radial equation of the reduced radial wave function in dimensionless units, and show that it must have the form,

\[
u_{n,l}(\xi) = r^{l+1} e^{-\xi^2 / 2} F_{n,l}(\xi),
\]

where \( F_{n,l}(\xi) \) is constant near the origin, and does not blow up faster than \( e^{\xi^2} \) for large \( \xi \).
(c) Show that in fact, the radial wave functions are,

\[ R_{nr,l}(r) = r^l e^{-r^2/2} L_{n-1/2}^l (r^2), \] (unnormalized)

where \( L_p^q(x) \) are the associated Laguerre polynomials.

\[ E_{nr,m} = \hbar \omega (2n_r + l + 3/2) = \hbar \omega (n + 3/2), \]

where the “principal quantum number” is defined by \( n = 2n_r + l \). Sketch the first three degenerate energy levels, and label the \( s,p,d \) states. Show again that the degeneracy of the energy eigenvalues you found are as in part (a).

**Problem 3: Hydrogenic atoms and atomic units.** (15 points)

Consider the “hydrogenic” atoms - that is bound-states of two oppositely charged particles:

(i) The hydrogen atom: Binding of an electron and proton.

(ii) Heavy ion: Single electron bound to a nucleus of mass \( M \), charge \( Z e \) (say \( Z=50 \)).

(iii) Muonium: Muon bound to a proton

(iv) Positronium: Bound state of an electron and a positron (anti-electron)

(a) For each, using the charges, reduced mass, and the unit \( \hbar \), determine the characteristic scales of:

- Length, energy, time, momentum, internal electric field, and electric dipole moment.

Please give numerical values as well as the expressions in terms of the fundamental constants.

(b) Now add the speed of light \( c \) into the mix. Find characteristic velocity in units of \( c \), magnetic field, and magnetic moment. Show that for the particular case of hydrogen the characteristic velocity is \( v/c = \alpha = \frac{e^2}{\hbar c} (\text{cgs}) \approx 1 / 137 \), the “fine-structure” constant, and that the Bohr radius, Compton wavelength, and “classical electron radius”, differ by powers of \( \alpha \) according to,

\[ r_{\text{class}} = \alpha \lambda_{\text{compton}} = \alpha^2 a_0 \]

(c) What is the characteristic magnetic field and magnetic dipole moment?